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The Tsallis Distribution and Generalised Entropy: Prospects for Future Research into Decision-Making under Uncertainty[1]

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1.0 Introduction

This paper reviews recent work on the properties of Tsallis distributions, Tsallis statistics, anomalous diffusion and multi-fractal processes. Processes of this kind have the capacity to represent natural phenomena that are associated with turbulence and emerging complexity. However, in addition they may also capture crucial aspects of human decision-making in response to conditions of uncertainty. This leads to the obvious question of the inter-relationship between each of these two fields: the behavioural and the empirical. The Tsallis or *q*-exponential distribution describes both an *Ontological* property (i.e. an empirical characterisation of existing stochastic processes such as wind turbulence, river flooding, rainfall, asset prices, transport through doped media, DNA sequencing, the arrival of news within information networks etc); and an *Epistemic* attribute (i.e. a behavioural characterisation of decision-making as seen in generalisations of subjective expected utility designed to account for investor aversion to ambiguity or uncertainty). The paper articulates the linkages between the property of non-extensivity possessed by generalised information measures such as Tsallis entropy, *S*-shaped distortion functions that are commonly applied in the actuarial sciences, coherent risk measures discussed in the finance literature, and what is called Choquet expected utility theory in the economics literature.

The dual characterisation of the Tsallis distribution raises broader questions about the linkage between both the ontological and epistemic domains. In previous research Juniper (2005) has attempted to explain this linkage by arguing that an increasing fragility of real world phenomena would give rise to a heightened aversion to ambiguity on the part of decision makers. This is because fragility implies an increased sensitivity of the economy to adverse conditions. It is this increase in sensitivity that leads to a heightening of uncertainty aversion on the part of decision makers. Arguably, this kind of ontologico-epistemic interaction could be responsible for many of the reported 'self-fulfilling prophecy' phenomena in both economics and finance. In this regard, the very nature of the stochastic processes governing risk would, themselves, influence these adverse conditions. For example, stochastic processes conforming to Tsallis Distribution, Mandelbrot's multifractal processes or Lévy-Stable Laws give rise to more extreme tail probabilities in comparison with Geometric Brownian motion, which would increase estimates of value-at-risk.

The next section of the paper examines the distinction between uncertainty and risk. It then considers the distinction post-Keynesian theorists have attempted to make between aversion to uncertainty and aversion to risk. In an effort to dig deeper into these distinctions, the next section of the paper reviews a comprehensive study by Van der Lubbe et al. (1984), which adopts an information theoretic approach to the derivation of generalised measures of uncertainty. Three families of uncertainty measures are constructed which encompass almost all of the information measures in common usage, including Tsallis entropy. The property of non-extensivity or pseudo-additivity is also introduced in this section of the paper. Section 3 of the paper provides an overview of the *q*-algebra that is associated with Tsallis entropy and the related Tsallis distribution. It reviews recent research, which has deployed the *q*-generalised exponential and logarithmic functions to establish a generalisation of the

Lévy-Gnedenko central limit theorem. Sub-section 3.1 draws from what is now an extensive literature, some of the more pertinent relationships between Tsallis entropy, statistical distributions, and stochastic processes. Section 4 of the paper returns to examines the property of non-extensivity, which is associated with the hyperbolic family of generalised uncertainty measures, including Tsallis entropy. While section 4.1 highlights the relationship between non-extensivity, Kahneman and Tversky's notion of Bounded Sub-additivity and Uncertainty Aversion in decision-making; section 4.2 examines the relationship holding between non-extensivity, Coherent Risk Measures, and the *S*-shaped distortion functions that arise in both the actuarial analysis of risk and in economic applications of Choquet expected utility theory. Section 5 of the paper interrogates the linkage between sub-additive probabilities and multiple-priors approaches to robust control and estimation. This sets the scene for a series of speculations on the nature of human cognition, the evolution of pattern-recognition capabilities, which respond to the fractal geometry of nature, and uncertainty aversion in human decision-making. These speculative forays informs the analysis of section 6, which draws together strands of analysis taken from other sections of the paper to embark on a Keynesian and Minskyian critique of quantitative finance theory. Concluding comments follow in section 7.

2.0 Uncertainty Aversion and Risk Aversion

In Book V of *The General Theory* Keynes argued against the then-dominant Treasury view that wage and price deflation would automatically cure any departures from full employment through increases in expenditure induced by falling real interest rates and continuously falling prices. Instead, Keynes emphasised the adverse impact of wage-price deflation on both the marginal efficiency of capital schedule, the precautionary demand for money, and the marginal propensity to consume (via shifts in income from high spending borrowers to low spending lender triggered by deflation under a regime of non-indexed, nominal contracting). However, the concerns of this paper are more broadly focused on the increased sensitivity of the economy (as reflected in the balance sheet positions of key decision-makers—households, banks and firms) to adverse changes in sentiment (i.e. increasing aversion to uncertainty or ambiguity).

While an increasing number of macroeconomists and quantitative finance theorists (including Hansen et al, 2002) are now willing to distinguish between aversion to risk and aversion to ambiguity or uncertainty, there are no doubt many who continue to hold to an outdated, rational expectations-informed position that the subjective probabilities reflecting aversion to risk could ultimately converge to objective probabilities. For the latter group of theorists, convergence would arise either through some kind of learning process, or rather through processes of evolutionary selection advantaging those agents who adopt more rational mechanisms in forming their expectations.

Amongst heterodox theorists a more commonly held position would be that comprehensive information that would enable economic agents to predict future by means of frequency distributions does not exist due to the inherent creativity and unpredictability of human interventions that can transform institutions, introduce new products and services, and develop new ways of doing things (Dequech, 2000). As such, over varying time horizons and in various combinations, innovative forms of social interaction would transform the stochastic processes characterising economic phenomena. Accordingly, any attempts by theorists to formalize models of uncertainty aversion, that could in turn inform policy interventions, are dismissed as the economic equivalent of “paddling a canoe with a butterfly net”.[2]

In contrast to such forms of heterodox skepsis, this paper sees virtue in the development of formal models of uncertainty aversion that could inform prudential control and policies designed to offset the damaging consequences of the business cycle. Accordingly, the next section of the paper begins with an outline of generalized information measures that could serve as an integrative framework for further inquiry into the nature of uncertainty aversion. It will be shown that these measures, while conforming to Kolmogorov’s axioms of probability, are sufficiently general to account for complex, multifractal stochastic processes.

2.1 Generalised Uncertainty Measures

In a discrete probability setting Van der Lubbe et al. (1984), demonstrate the existence of three families of information measures based on the relation between information and *certainty* rather than on the conventional mode of derivation, which is based on the relation between information and *uncertainty*. Thus, each information measure is constructed from two certainty measures, each of which conforms to a set of desirable formal properties. From the resulting certainty measure the authors derive a strictly monotonic, continuous measure of average certainty. This enables the authors to derive (van der Lubbe et al., 1984, definition 2: 197) three general classes of information measures for parameters $(\rho, \sigma, \delta) \in E = \{\{\rho, \sigma, \delta \mid (\rho, \sigma) \in D \wedge \delta > 0\}$:

a) the logarithmic information measure given by,

$$ {}^{1}H_n(P;\rho,\sigma,\delta) = -\delta \cdot \log_2 G_n(P;\rho,\sigma) = -\delta \cdot \log_2 \left[\sum_{i=1}^{n} p_i^{\rho} \right]^{\sigma}, $$

b) the linear information measure given by,

$$ {}^{2}H_n(P;\rho,\sigma,\delta) = \delta\left[1 - G_n(P;\rho,\sigma)\right] = \delta\left\{1 - \left[\sum_{i=1}^{n} p_i^{\rho} \right]^{\sigma} \right\}, \text{ and} $$

c) the hyperbolic information measure given by,

$$ {}^{3}H_n(P;\rho,\sigma,\delta) = \delta\left[\frac{1}{G_n(P;\rho,\sigma)} - 1 \right] = \delta\left\{ \left[\sum_{i=1}^{n} p_i^{\rho} \right]^{-\sigma} - 1 \right\}. $$

It can be seen that each of these measures operates with a different formal representation of uncertainty, where the latter defined in relation to average certainty (i.e. the summation term appearing in the square brackets and raised to the power of

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[2] This metaphor was conveyed to the author in an email by Professor Paul Davidson in a critique of the author’s earlier attempts to develop a formal model of Keynesian liquidity preference.

σ). Next, van der Lubbe et al. (1984, theorem 4: 201-202) establish conditions that such information measures, $H(X)$, should satisfy—namely: non-negativity, continuity, strict monotonicity $H(P) = I[f(P)] \geq 0$ of the information measure; and for stochastically independent experiments P and Q, that $H(PQ) = H(P) + H(Q) + c\, H(P)\cdot H(Q)$—which enables them to determine non-trivial solutions for the measure:

$c = 0$ for ${}^1H_n$, with $\rho = ab + 1$, $\sigma = 1/b$, and $(\rho, \sigma, \delta) \in E$;

$c < 0$ for ${}^2H_n$, with $\delta = -1/c$, $\rho = ab + 1$, $\sigma = 1/b$, and $(\rho, \sigma, \delta) \in E$;

$c > 0$ for ${}^3H_n$, with $\delta = 1/c$, $\rho = ab + 1$, $\sigma = d/b$, with $d > 0$, and $(\rho, \sigma, \delta) \in E$.

Moreover, van der Lubbe et al (2001: 204) show that the following relations obtain between each of the measures,

$$
\begin{aligned}
{}^1H_n(P;\rho,\sigma,\delta) &= -\delta \log_2\left[1 - \frac{{}^2H_n(P;\rho,\sigma,\delta)}{\delta}\right], \\
{}^2H_n(P;\rho,\sigma,\delta) &= \frac{\delta\, {}^3H_n(P;\rho,\sigma,\delta)}{{}^3H_n(P;\rho,\sigma,\delta)}
\end{aligned}
$$

A variety of minor theorems and corollaries then establish the properties of these information measures. ${}^1H_n$ is Rényi entropy with $\rho = \alpha$, $\sigma = 1/(\alpha - 1)$, $\delta = 1$. ${}^2H_n$ is Havrda and Charvat or Daroczy entropy for $\rho = \beta > 0$, $\beta \neq 1$, $\sigma = 1$, $\delta = 1/(1 - 2^{1-\beta})$. The Sharma and Mittal Information measure of order α and type β results for ${}^2H_n$ with $\rho = \alpha$, $\sigma = (\beta - 1)/(\alpha - 1)$, $\delta = 1/(1 - 2^{1-\beta})$. Arimoto's R-norm results when $\rho = R$, $\sigma = 1/\rho$, $\delta = R/(R-1)$. Lansberg's entropy measure results from ${}^2H_n$ when $\rho = q$, $\sigma = -1$, and $\delta = 1/(1-q)$. Finally, Tsallis entropy can be derived from van der Lubbe et al's hyperbolic ${}^3H_n$ information measure through the substitutions: $\rho = q$, $\sigma = -1$, $\delta = 1/(1 - q)$, as noted by Tsallis (1995).

From the second class of certainty measures the authors derive two additional families of information measures given by,

$$
\begin{aligned}
H(X) &= da\sum_{i=1}^{n} p_i \log p_i, \quad a > 0,\ d < 0; \\
H(X) &= \frac{1}{c}\left\{\left[\prod_{i=1}^{n} p_i^{p_i}\right]^{ad} - 1\right\}, \ a > 0,\ dc < 0
\end{aligned}
$$

The first of these is identical to that of Shannon Entropy up to a constant, while the second includes Leti's measure of relative diversity.

Van der Lubbe et al's (2001: 194) paper has noteworthy implications. Associated with the three families of information statistics are three related forms of expectation operator. A unique version of probabilistic analysis can be constructed for each class of expectations operator without the necessity for any resort to measure theory. Accordingly, the theorist is able to choose the particular class of analysis that is most appropriate for his or her specific inquiry. One example of this is the class of probability theory pertaining to non-extensive statistical mechanics. Non-extensivity, often termed pseudo-additivity, can take a super- or sub-additive form. It is defined by non-zero values of *c* in the relationship that is presumed to obtain between information measures defined over statistically independent distributions *P* and *Q*. This relationship (reproduced for convenience below) relates the measure of the joint distributions to their individual and marginal distributions,

$H(PQ) = H(P) + H(P) + \mathrm{c}\, H(P) \cdot H(Q)$.

Here the *c* parameter embodies the property of global correlation between independent distributions. In the development of non-extensive statistical mechanics both Rényi and Tsallis entropy have played a dominant role. The following section of the paper first discusses the properties of Tsallis entropy, its supporting axioms, and their relationship to Lévy processes, before examining the implications of the property of pseudo-additivity for decision-making under uncertainty.

3.0 The *q*-algebra and the *q*-generalised Central Limit Theorem

In explicating the properties of what is now commonly referred to as the *q*-algebra, Suyari (2004) reveals what he describes as the underlying beauty and simplicity of the mathematical processes that result in the Tsallis distribution. Suyari (2004:2) begins by defining the *q*-product as follows,

$$x \otimes_q y := \begin{cases} \left[x^{1-q} + y^{1-q} - 1\right]^{\frac{1}{1-q}}, & \text{if } x > 0, y > 0, x^{1-q} + y^{1-q} - 1 > 0 \\ 0, \text{ otherwise} & \end{cases}$$

Following Tsallis (1994), Suyari introduces the *q*-logarithmic and *q*-exponential functions defined below,

$$\ln_q(x) := \frac{x^{1-q} - 1}{1-q} \quad x > 0, q \in \Re^+$$

$$\exp_q(x) := \begin{cases} \left[1 + (1-q)x\right]^{\frac{1}{1-q}} & \text{if } 1 + (1\text{-}q)x > 0,\ x > 0, q \in \Re^+ \\ 0 & \text{otherwise} \end{cases}$$

On the basis of these definitions the *q*-product satisfies the following:

$$\ln_q(x \otimes_q y) = \ln_q x + \ln_q y$$

$$\exp_q(x) \otimes_q \exp_q(y) = \exp_q(x + y)$$

The *q*-exponential function can be derived in a manner analogous to that for the conventional exponential function (i.e. $\exp(x) = \lim_{n \to \infty}\left(1 + \frac{x}{n}\right)^n$) using the following limit expression (Suyari, 2004: 3):

$$\exp_q(x) = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)\otimes_q^n .$$

The q-sum of two real numbers, which is commutative, associative, and recovers the conventional summation operation when $q = 1$, is defined by the following,

$$x \oplus_q y = x + y + (1-q)xy \ .$$

By inversion, q-subtraction, which is also commutative, associative, and recovers the conventional summation operation when $q = 1$, can be defined as,

$$x(-)_q\, y = \left[x^{1-q} + y^{1-q} - 1\right]^{\frac{1}{1-q}}$$

This q-algebra plays a crucial role in the burgeoning field of Tsallis statistics. For example, Suyari and Tsukada (2005) demonstrate that the Tsallis distribution,

$$f(x) = \frac{\exp_q\left(-\beta_q x^2\right)}{\int \exp_q\left(-\beta_q x^2\right)dx},\ \ \beta_q > 0,$$

can be derived by taking the maximal value of the q-product of the likelihood function, $L_q\,(\theta)$, shown below,

$$L_q(\theta) = L_q(x_1, x_2, \ldots, x_n; \theta) := f(x_1-\theta)\otimes_q f(x_2-\theta)\otimes_q \cdots \otimes_q f(x_n-\theta) \text{ at}$$

$$\theta = \theta^* := \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

Suyari shows that the q-product can also be applied in deriving the q-Sterling's formula for the q-factorial $n!_q$ for $n \in \mathbf{N}$ and $q > 0$ (Suyari, 2004:5),

$$n!_q := 1\otimes_q \cdots \otimes_q n = \left[\sum_{k=1}^{n} k^{1-q} - (n-1)\right]^{\frac{1}{1-q}}.$$

The Central Limit Theorem (CLT) implies that any sum of N *independent* random variables will tend, as $N \to\infty$, to be distributed according to a certain law (which operates as an attractor in the space of distributions. When the distribution of the individual random variables has *finite* variance the asymptotic distribution for the sum will be the Normal or Gaussian distribution. In the *de Moivre-Laplace theorem* it is demonstrated that asymptotically, as $N \to\infty$, the Binomial distribution approaches a Gaussian distribution. The Lévy-Gnedenko-Kolmogorov generalisation of the CLT states that the asymptotic distribution of the sum of *N independent, infinite*-variance random variables will be the Lévy distribution. Suyari (2004) deploys his q-logarithmic generalization of Sterling's formula to establish numerical indications of the limiting properties of generalized q-binomial and q-multinomial models, showing that in each case, they converge to the Tsallis distribution.

In the literature on a q-algebra there is a natural progression from the q-arithmetic (the q-sum, q-subtraction, q-product and q-division) through to the hyperbolic functions such as the q-logarithm and q-exponential. It is then only a small step to the construction of a q-generalisation of the Fourier transform. It is this version of the

Fourier transform, that Umarov, Tsallis, Gell-Mann and Steinberg (2006a,b) deploy to accomplish their generalization of the Lévy-Gnedenko central limit theorem. An approach of this kind was inevitable given earlier research highlighting numerical indications of a *q*-generalised central limit theorem, and the fact that the original Lévy-Gnedenko version of the central limit theorem was originally conceived and executed entirely within the frequency- rather than the time-domain using the Fourier transform.

3.1 Tsallis Entropy and Stochastic Processes

While Boltzmann-Shannon-Gibbs (BSG) entropy is defined by (Abe, 2000):

$$S(p_1, p_2, \cdots, p_W) = -k\sum_{i=1}^{W} p_i \ln p_i \ ,$$

Tsallis entropy is defined by:

$$S_q(p_1, p_2, \cdots p_W) = \frac{1}{1-q}\left[\sum_{i=1}^{W}(p_i)^q - 1\right](q > 0),$$

In this generalization of BSG-entropy, the *q* parameter represents the degree of nonextensivity. Tsallis's nonextensive measure meets Kapur and Kesavan's requirements for a *generalized measures of cross entropy* (1994:309-15). In section 2.1 it was shown that Tsallis entropy could be derived from van der Lubbe et al's hyperbolic ${}^3H_n$ information measure through the substitutions: $\rho = q$, $\sigma = -1$, $\delta = 1/(1 - q)$, as noted by Tsallis (1995).

Tsallis entropy provides a useful ansatz for the calculation of solutions to certain nonlinear partial differential equations. Moreover, under appropriate constraints the maximization of Tsallis entropy also yields exact time-dependent solutions for a family of non-linear Fokker-Planck equations representing anomalous diffusion and certain self-organizing phenomena (Tsallis, 1995; Tsallis et al., 1998). These equations are characterized by a diffusion term depending on the power of the probability density. Under equiprobability,

$$S_q = \max S_q = k \ln_q W.$$

In the 20s and 30s, Lévy was concerned with the question of how to represent a situation of fractal scaling where the sum of identically random distributed variables has the same probability distribution as any one of the terms in the sum. The resulting distributions are now called Lévy's stable laws (Shlesinger et al., 1987, p. 1100). Drawing on initial work by Sainty (1992), Jumarie (2000, Chapters 6 and 7) sets out a mathematically simpler construction of complex-valued fractional Brownian motion (C-$(fBm)_n$), conceived as the limit of random walks in the complex plane.

In contrast to a conventional random walk, for which large step lengths are (exponentially) rare, a Lévy flight is a random walk whose step length occurs with a power law frequency (Gupta & Campanha, 2002, p. 531). Thus Lévy flights have infinite variance. In real systems the variance of a stationary process is finite. Therefore, to describe such systems using Lévy flight processes, some kind of arbitrary cut-off must be imposed. Early research in this vein by Mantegna and Stanley (1995) deployed a truncated Lévy flight process. More recent developments (Gupta and Campanha, 2000, 2002) allow for the gradual elimination of large step sizes by using an exponential, capacity-related, cut-off term. The resulting gradually truncated Lévy distributions (GTLDs) approach the Gaussian distribution at relatively low-frequencies, but at high frequencies gives rise to a power-law distribution.

Montroll and Schlesinger (1983: 215) have shown that Lévy processes can be derived from maximizing Shannon-Boltzmann entropy under the usual normalisation condition if the following moment constraint is imposed,

$$H = -\int_{-\infty}^{\infty} p(x) \log\left(\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixk} e^{-|k|^{\alpha} a} dk \right) dx = \text{constant.}$$

However, they acknowledge that “it is difficult to imagine that anyone in an *a priori* manner would introduce” a condition of this nature for maximising entropy[3]. Indeed, Tsallis (1995) uses this presumption to justify his favoured approach based on generalised entropy.

While GTLDs are based on positive feedback and physical limitations, Tsallis statistics are based on generalized thermodynamic considerations. Nevertheless, both statistics yield almost the same distribution. For this reason, Gupta and Campanha (2002, p. 385) speculate that the parameters of the GTLD are related to the q parameter because the limit that arises is due to similar thermodynamical or other natural requirements.

For statistically independent systems A and B, under Tsallis entropy, it is well known that *pseudoadditivity* obtains (which is congruent with the findings of Lubbe et al, 2001, discussed in section 2.1 of the paper), namely:

$S_q[A, B] = S_q[A] + S_q[B] + (1-q)S_q[A]S_q[B]$ (see Di Sisto et al, 1999)[4].

Another characterization, which obtains for a partition of this probability space into two segments, one for low valued probabilities (summing to p_L) and the other for high valued probabilities (summing to p_M), of pseudoadditivity for systems A and B is:

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[3] This moment constraint is related to the Fourier transform of the Lévy distribution which reads,

$$p(\mathbf{k}) = \int \exp(i\mathbf{k} \cdot \mathbf{x}) p(\mathbf{x}) d\mathbf{x} = \exp(-bk^{\gamma}),$$

where b is a positive constant and $k \equiv |\mathbf{k}|$. It is this transform, which Lévy used to define the distribution named after him, that is responsible for the self-similarity property of the distribution, because it converts the convolution of two Lévy distributions with the same exponent into a third Levy distribution with the same exponent (Zanette, 1999):

$$p_1(k)p_2(k) = \exp(-b_1 k^{\gamma})\exp(-b_2 k^{\gamma}) = \exp[-(b_1 + b_2)k^{\gamma}] = p_3(k).$$

[4] Compare this result with van der Lubbe et al’s theorem four.

$$S_q(p_1, p_2, \cdots, p_W) = S_q(p_L, p_M) + (p_L)^q S_q \times (p_1|p_L, p_2|p_L, \cdots, p_W|p_L)$$
$$+ (p_M)^q S_q \times (p_1|p_M, p_2|p_M, \cdots, p_W|p_M).$$

where, $p_L = \sum_{i=1}^{W_L} p_i$, $p_M = \sum_{i=W_L+1}^{W} p_i$, with $W_L + W_M = W$, and the respective sets $\{p_i|p_L\}$.and $\{p_i|p_M\}$.are conditional probabilities. Because $p_i^q > p_i \leftrightarrow q < 1$, $p_i^q < p_i \leftrightarrow q > 1$, superadditivity can be seen to privilege *rare* events while subadditivity privileges *frequent* events (Tsallis et al, 1998, p.535). Although pseudo-additivity is familiar to theoretical physicists, important applications have also arisen in quantitative finance and the theory of decision-making under uncertainty (see Schmeidler, 1989). These applications will be examined in section 4 of the paper.

Abe (2000) shows how the Shannon-Khinchin axioms for Boltzmann-Shannon entropy can be modified to accommodate Tsallis entropy. His paper establishes that a quantity satisfying the transformed axioms is uniquely equal to Tsallis entropy. Thus, his uniqueness result represents a natural generalization of the Shannon-Khinchin result for Boltzmann-Shannon entropy by establishing a parallelism with the original axioms. Tsallis et al (1995) argue that the ubiquity and robustness of the Lévy distribution follow naturally from the generalized central-limit theorem, which applies to convolutions of distributions. Significantly, they further demonstrate that that Tsallis entropy generalizes the traditional inverse relationship known to hold between Boltzmann-Shannon entropy and the exponential function.

Abe (1997) shows that Tsallis entropy can be interpreted using Jackson's generalized differential operator. While Jackson's operator "tests" the function $f(x)$ under *dilation*, the usual derivative tests it under *translation*. This feature explains the usefulness of Tsallis entropy for describing chaotic systems with multifractal characteristics[5]. Under appropriate moment constraints over the first and second moments of the distribution, Boltzmann-Shannon entropy can be used to derive the familiar Gaussian process. However, under slightly modified moments constraints (which take into account the divergence of the second moment), De Souza and Tsallis (1997) also show that Tsallis entropy can be used to derive the Students-*t* distribution. Similarly,

[5] Jackson's generalized differential operator, D_q, defined for an arbitrary function $f(x)$ is given by:

$$D_q f(x) \equiv \frac{f(qx) - f(x)}{qx - x}.$$

Tsallis et al (1998, p. 536) and Abe (1997) observe that while Boltzmann-Gibbs entropy satisfies the following relation:

$$-k\left[\frac{d}{d\alpha}\sum_{i=1}^{W} p_i^{\alpha}\right]_{\alpha=1} = -k\sum_{i=1}^{W} p_i \ln p_i \equiv S_1,$$

Tsallis entropy satisfies the relation:

$$-k\left[D_q \sum_{i=1}^{W} p_i^{\alpha}\right]_{\alpha=1} = k\frac{1-\sum_{i=1}^{W} p_i^q}{q-1} \equiv S_q.$$

Abe and Turner (2005) show how the assumptions made by Einstein in his classic derivation of Brownian motion can be relaxed (specifically, the assumption relating to the existence of the second moment of the distribution is replaced by one assuming that the distribution has a divergent second moment whose characteristic function is given by a 'stretched exponential form'), so that the solution to the diffusion equation (obtained using the techniques of fractional calculus) meets the defining characteristics of the Lévy distribution.

4.0 Non-extensivity

This section of then paper re-examines the property of non-extensivity, which is associated with the hyperbolic family of generalised uncertainty measures, including Tsallis entropy. Specifically, section 4.1 highlights the relationship between non-extensivity, Kahneman and Tversky's notion of Bounded Sub-additivity and Uncertainty Aversion in decision-making. Section 4.2 examines the relationships between non-extensivity, Coherent Risk Measures, and the *S*-shaped distortion functions that arise in both the actuarial analysis of risk and in economic applications of Choquet expected utility theory.

4.1 Bounded Sub-additivity and Uncertainty Aversion

Significantly, Tsallis et al (2003) comment on the relationship between the property of pseudoadditivity and *Cumulative Prospects Theory* (CPT)—Kahneman and Tversky's model of non-expected utility. The authors also cite Dow and Werlang's work on Choquet Expected Utility Theory in Tsallis (1995)[6].

Anteneodo and Tsallis (2003) acknowledge this generalization of Prospect theory to a rank-dependent utility form, which entails an *S*-shaped distortion of the cumulative distribution function. However, because their paper only considers simple prospects with a single positive outcome, the specific role of Choquet Integration is not clarified or expounded. This simplification allows them to consider a variety of straightforward functional forms in calculating *q*-expectation values, including:

$$\Pi_1(p)=\frac{p^q}{p^q+(1-p)^q};\ \ \Pi_2(p)=\frac{p^q}{\left[p^q+(1-p)^q\right]^{1/q}},\ \ \Pi_3(p)=\frac{p^q}{\left[p^q+A(1-p)^q\right]},\ A>0.$$

Queirós, Anteneodo, and Tsallis (2005: equations 5 and 6) note that similar functional forms appear in the moment constraints for the generalized mean and generalized variance, which are imposed when Tsallis entropy is maximized to yield the *q*-Gaussian PDF[7]. The second and third of these forms are identical to the *S*-shaped PWFs appearing in Cumulative Prospect Theory (also see Prelec, 1998, equations 3.5,

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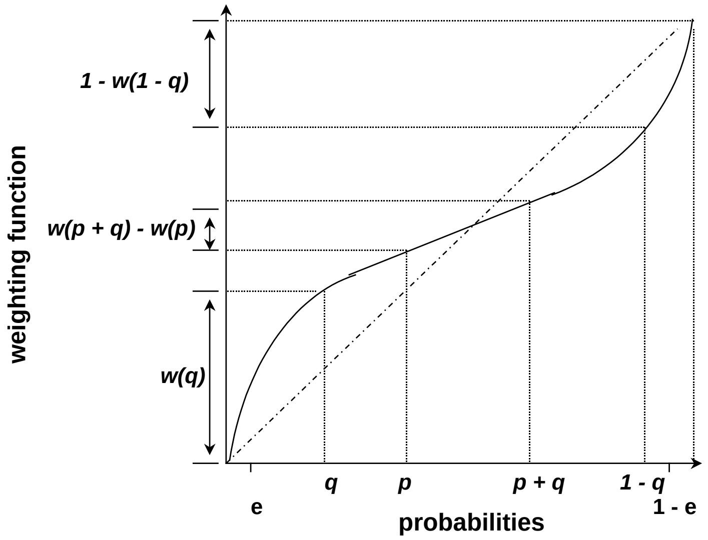
[6] However, in his 1995 paper Tsallis warns that Choquet's mean value for a positive real constant λ, would yield λ, whereas the generalized Tsallis version of mean-value would result in a value generically smaller than λ for values of $q > 1$.

[7] $\langle x\rangle_q=\bar{\mu}_q=\int x\frac{[p(x)]^q}{\int[p(x)]^q dx}dx;\ \left\langle\left(x-\bar{\mu}_q\right)^2\right\rangle_q=\bar{\sigma}_q^2=\int\left(x-\bar{\mu}_q\right)^2\frac{[p(x)]^q}{\int[p(x)]^q dx}dx$

3.6: 506). Tversky and Wakker's straightforward interpretation of this weighting function is illustrated below. Bounded sub-additivity obtains for the weighting function, *w*, if there exist boundary constants $\varepsilon \geq 0, \varepsilon' \geq 0$ such that:

$$w(q) \geq w(p+q) - w(p) \text{ whenever } p+q \leq 1-\varepsilon, \text{and}$$
$$1 - w(1-q) \geq w(p+q) - w(p) \text{ whenever } p \geq \varepsilon'$$

Cumulative Prospect Theory and Subadditivity in Preferences



The first of these conditions—*upper sub-additivity*—implies that a shift in probability has more impact when it makes an event certain than when it makes an event more probable. The second of these conditions—*lower sub-additivity*—implies that a shift is probability has more impact when it makes an event possible than when it merely increases the probability of an event. In the generalized model of uncertainty aversion, the property of *bounded subadditivity* implies that an event has a greater impact when it turns impossibility into possibility or possibility into certainty, than when it merely makes a possibility more likely (Tversky and Wakker, 1995, p. 1264).

Queirós, Anteneodo, and Tsallis go on to demonstrate that *q*-Gaussian distributions can also be derived from (microscopic dynamic) stochastic processes characterised by multiplicative noise,

$$\dot{x} = f(x) + g(x)\zeta(t) + \eta(t),$$

stochastic processes of linear form,

$$\dot{x} = -\gamma x + x\zeta(t) + \eta(t),$$

and stochastic processes with varying intensive parameters. Queirós (2005) examines a model of high-frequency stock trading volume with a stationary PDF given by the following *q*-exponential function:

$$p(\upsilon) = \frac{1}{Z}\left(\frac{\upsilon}{\theta}\right)^{\alpha} \exp_q\left(-\frac{\upsilon}{\theta}\right) .$$

Here υ represents traded volume as a normalized ratio of mean trading volume while α and θ are positive parameters. Meanwhile, it is presumed that trading volumes are governed by the following mean-reverting stochastic differential equation,

$$d\upsilon = -\gamma\left(\upsilon - \frac{\alpha+1}{\beta}\right)dt + \sqrt{2\upsilon\frac{\gamma}{\beta}}dW\ ,$$

with W_t representing a zero mean and unitary variance Wiener process. Queirós shows that the stationary distribution is a member of the Gamma family with mean value $<\upsilon> = (1 + \alpha)/\beta$, and standard deviation $< \upsilon - <\upsilon>>^2 = (1 + \alpha)/\beta^2$. This conventional model is modified by assuming that β follows a (stationary) Gamma PDF,

$$P(\beta) = \frac{1}{\lambda\Gamma[\delta]}\left(\frac{\beta}{\lambda}\right)^{\delta-1}\exp\left(-\frac{\beta}{\lambda}\right),\quad \delta > 0,\ \lambda > 0\,.$$

This transforms the PDF for the marginal distribution of υ into what Queirós calls a q-generalized Gamma probability function,

$$P(\upsilon) \equiv \frac{(q-1)^{\alpha+1}\Gamma\left[\frac{1}{q-1}\right]}{\theta\Gamma\left[\frac{1}{q-1}-\alpha-1\right]\Gamma[\alpha+1]}\left(\frac{\upsilon}{\theta}\right)^{\alpha}\exp_q\left(-\frac{\upsilon}{\theta}\right),\quad \lambda = \frac{q-1}{\theta},\ \delta = \frac{1}{q-1}-\alpha-1.$$

When $q \to 1$, implying the absence of fluctuations in β so that $P(\beta)$ becomes a delta function centred in θ^{-1}, the traditional Gamma PDF is recovered. For small values of υ, $P(\upsilon) \sim \upsilon^{\alpha}$. However, for large values of υ, $P(\upsilon) \sim \upsilon^{\alpha/(1-q)}$. Through numerical simulation of this distribution Queirós shows that it closely matches empirical data (relating relative frequency × volume of trading activity) for ten high-volume stocks traded on NASDAQ. Queirós interprets this model to be one characterised by two ingredients: macroscopic memory (represented by multiplicative noise) and microscopic herding by traders (represented by local temporal fluctuations in β or the mean value of υ caused by rumours, news, and price movements)[8].

Queirós and Tsallis (2004) examine the equivalence between second and fourth moments calculated for certain parameterisations of the q-Gaussian distribution and those derived from Engel's ARCH(1,1) model of stochastic volatility. Queirós and Tsallis (2005) extend these results to incorporate Engel's GARCH(1,1) process. Using a q-generalized form of Kullback-Leibler relative entropy, they examine temporal dependence between successive returns for the GARCH(1,1) process.

Nevertheless, in the absence of behavioural interpretations, the whole exercise of representing financial returns or volumes by q-generalised conditional distributions amounts to an elaborate process of 'curve-fitting'. In identifying the link between Tsallis entropy and Cumulative Prospect Theory, Tsallis and his colleagues failed to relate it to the axioms of Choquet utility theory, which are responsible for the

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[8] For an alternative phenomenological model applied to closed-form pricing of options incorporating skewness and smile see Borland (19989, 2002) and Borland and Bouchaud (2004).

resulting *S*-shaped distortion functions. No doubt, a set of behavioural axioms similar to those constructed for Cumulative Prospect Theory will soon be derived for financial applications of Tsallis entropy. This task has already been accomplished for Shannon-Boltzmann entropy, whose relationship to exponential utility functions is now well understood. All that has been achieved so far is recognition of the *analogies* holding between pseudo-additivity for Tsallis entropy, and bounded subadditivity for decision-making under uncertainty. These linkages are examined in the next section of the paper.

4.2 Non-extensivity, Coherent Risk Measures, Choquet expected utility theory, and *S*-shaped distortion functions

Kapur and Kesavan's Entropy Optimization Postulate (1992: 297) suggests that every probability distribution, theoretical or observed, is an entropy optimisation distribution (i.e. it can be obtained by minimizing a cross-entropy measure with respect to an appropriate prior distribution, subject to appropriate moment constraints). The relationship between minimum relative entropy and the Exponential Family of distributions is well known (Reesor and McLeish, 2002: 18-19). In addition, relative entropy is closely related both to sub-additivity and Choquet integration (Reesor and McLeish, 2002) and Mirofushi and Sugeno (1989) discuss the link between Fuzzy measure theory and Choquet integration.

In the actuarial sciences *distortion measures*, which have a Choquet Integral representation, are widely used to determine insurance premium risk (Wang, 1996a,b). A distortion function, g, is any non-decreasing function on [0,1] such that $g(0) = 0$ and $g(1) = 1$. If a random variable X under the probability measure P has a cumulative distribution function (cdf) F defined by $F(x) = P[X \leq x]$, and a decumulative distribution function (dff) S defined by $S(x) = P[X \geq x] = 1 - F(x\text{-})$, and if $g(u)$ is a left-continuous distortion function then $S^{*}(x) = g[S(x)]$ is a ddf corresponding to a distorted probability distribution. The dual distortion function is given by $\bar{g}(u) = 1 - g(1-u)$.

In a comprehensive paper, Reesor and McLeish (2002: 16) bring together a range of properties relating to maximum entropy distributions. Citing earlier work by Dennenberg (1994: Chpts 5, 6), they set out axioms that enable them to derive the Choquet integral defined with respect to a distortion function. Following research by Wirch and Hardy (2000), Reesor and McLeish (2002:19; proposition 7, corollaries 8, 9) further demonstrate the precise relationship holding between: (a) the properties of the Choquet integral (specifically, sub-additivity and boundedness below the mean); (b) the non-positivity conditions that must be imposed on the moment constraints of a prior distribution when minimizing relative entropy; and, (c) the concavity properties of the distortion function. Citing Artzner, Dellbaen, and Heath (1999), Reesor and McLeish (2002: 21; definitions 7, 8, and 9) also demonstrate that the properties of *coherent risk measures*—namely; monotonicity, positive homogeneity, translation invariance, and sub-additivity—are precisely those satisfied by the Choquet integral. The necessary implication of this demonstrated relationship is that a risk measure is a *distorted* risk measure if it has a Choquet integral representation (Reesor and McLeish, Theorem 5, 2002: 15).

The use of the dual distortion function means that the distortion measure can be applied to the cdf to obtain $F^*(x) = \bar{g}[F(x)]$. Given a left-continuous distortion function $g(u)$ such that $S^*(x) = g[S(x)]$, or equivalently, $F^*(x) = \bar{g}[F(x)]$, then the Choquet Integral Distortion function can be derived by first noting that,

$$\begin{aligned} E_{P*}[X] &= E_{P*}[X^+] - E_{P*}[X^-] = \int_0^\infty S^*(x)dx - \int_0^\infty F^*(-x)dx \\ &= \int_0^\infty g(S(x))dx - \int_0^\infty \bar{g}(F(-x))dx = \int_0^\infty g(S(x))dx - \int_0^\infty [1 - g(S(-x))]dx \\ &= \int_0^\infty g(S(x))dx + \int_{-\infty}^0 [g(S(x)) - 1]dx \end{aligned}$$

Here $X^+ = \max(X, 0)$ and $X^- = \max(0, -X)$.

For any random variable X with ddf $S(x)$, the Choquet Integral with respect to a distortion function g is accordingly given by $H_g(\mathrm{X})$ in the following (Reesor and McLeish, 2002, definition 5:15),

$$\begin{aligned} H_g(X) &= \int_0^\infty g[S(x)]dx + \int_{-\infty}^0 [g[S(x)] - 1]dx \\ &= \int_0^\infty [1 - \bar{g}[F(x)]]dx - \int_{-\infty}^0 \bar{g}[F(x)]dx \end{aligned}$$

This raises the obvious question of how coherent or distorted risk measures (particularly those depending on the positive part of the loss given by $H_g(X^+)$), which have demonstrated equivalences to the Choquet integral and to specific forms of moment constraints that are imposed in relative entropy optimisation problems, can reflect behavioural attitudes to risk and uncertainty. Following Wirch and Hardy (2000), Reesor explains this relationship by defining a utility function $u(y) = -yg'(S(-y))$, which enables him to characterise the expected utility as,

$$E(u(-X^+)) = -\int_0^\infty (-x)g'(S(x))f(x)dx = \int_0^\infty xg'(S(x))f(x)dx = -H_g(X^+).$$

That is, the expected utility is the negative of the risk. Thus, distortion function can represent preferences towards risk or uncertainty. Moreover, the implied utility function u can be seen to depend on both the distribution S and the distortion function through the density $g'(S(\mathrm{x}))$, which describes how much the "risk-neutral" utility $u(x) = \mathrm{x}$ is modified by the distortion (Reesor: 17). This characterisation provides the necessary link to the finance literature on the determination of equivalent martingale measures when pricing assets in incomplete markets using relative entropy or generalised Esscher transforms (Stutzer, 1995; Chan and van der Hoek, 2001)[9]. Once

[9] Stutzer (1995, pp. 376-378) examines the relationship between minimum relative entropy, Gibbs state price probability densities, equivalent martingale measures, optimal portfolios associated with

again, the link between distortion functions and the distorted probability density gives us the necessary relation to the derivative of the distortion function, as in,

$$f^*(x) = -\frac{d}{dx}S^*(x) = -\frac{d}{dx}g(S(x)) = f(x)g'(S(x))$$
$$= -\frac{d}{dx}\overline{g}(S^*(x)) = \frac{d}{dx}\overline{g}(F(x)) = f(x)\overline{g}'(F(x))$$

Prelec (1998: 515) has shown that a preference relation satisfying the axioms that are sufficient for a Rank- and Sign-dependent representation of utility can, in turn, be represented by an S -shaped weighting function possessing the form [10],

$$\upsilon(p) = \gamma \exp\left\lfloor -\beta(-\ln p)^{\alpha}\right\rfloor.$$

If additional axioms of diagonal concavity, sub proportionality and compound invariance are satisfied, the specification takes the form[11],

$$\upsilon(p) = \exp\left\lfloor -(-\ln p)^{\alpha}\right\rfloor.$$

Groes et al., (1998) introduce two general and parsimonious axiomatic characterisations of the Choquet integral based on only two axioms, which are, respectively: the stochastic dominance axiom, and a minimum axiom. The first axiom requires that for any function, the integral with respect to a particular capacity should be larger than the integral with respect to another capacity, if the cumulative distribution derived from the function and the first capacity stochastically dominates the distribution derived from the first capacity. The second axiom, which accords with the properties of the ordinary integral for additive measures, requires that the integral with respect to a capacity assigning the value one to a specific set, and zero to all other sets, should be equal to the minimum of the integrated function over this set. Their analysis is only applied over finite sets, but the authors note that it could readily be extended to general sets through the introduction of a continuity axiom.

preferences that are represented by a constant absolute risk aversion utility function. He also provides three additional non-utility theoretic interpretations of the resulting state price density function based on quasi-maximum likelihood, a minimum information bound, and a Bayesian interpretation, which is related to Laplace's principle.

[10] Specifically, Prelec draws on axioms set out in Wakker (1994) and results from Wakker and Tversky (1993). Also, see Verlaine (2003: 9) and Miyamoto and Wakker (1996). Verlaine (2003) draws on Reesor and McLeigh's demonstration that a risk measure is a distorted risk measure if it has a Choquet integral representation (Reesor and McLeigh, 2002:21; definitions 7,8,9) to show that an *S*-shaped Probability Weighting Function (PWF) of the kind advocated in Choquet Expected Utility Theory is one consistent with maximising entropy, subject to a specific constraint defined over a measure of information. A paper by van der Hoek and Sherris (2001) completes the circle in showing how the distortion function approach based on a transformed hazard function can be modified to differentially treat upside and downside risk. Their chosen distortion function replicates the valuation of prospects under Yaari's (1987) dual theory of choice.

[11] See Tversky and Kahnemann, 1992 (cited in Prelec, 1998: 498).

5.0 Robust Control Theory and Human Cognition

While extensions of Cumulative Prospect Theory account for uncertainty through sub-additivity, an alternative perspective familiar to control theorists is one predicated on a multiple-priors approach. In a multiple-priors context, uncertainty aversion obtains when an agent's probabilistic beliefs are given by a *set* of probability measures rather than a singleton distribution. In characterizing the optimal rules in this context, researchers assume that economic agents adopt an intertemporal max-min expected utility approach: in a game–theoretic context, nature is presumed to be malicious in maximizing a penalty function through the choice of a particular probability density from within the range of permissible distributions. The agent is then presumed to minimize the same penalty function through the choice of a (sub-optimal) control law and filter (Petersen, James, and Dupuis, 2000; Elliott et al., 1995; Andersen, Hansen and Sargent, 1999). These rules are designed to protect the agent against unfavourable probabilistic structures in the financial environment. In this control theoretic context, the duality between free energy and relative entropy applies to the stochastic uncertainty constraint, which in turn accounts for (multiplicative) model uncertainty, observation error, and (typically non-Gaussian) perturbation. Gilboa and Schmeidler (1989) have established the mathematical equivalence between each of these capacity-based representations of uncertainty aversion: the first entailing the use of *subadditive probabilitie*, and the second deploying min-max optimization within a *multiple-priors* setting.

By drawing on white-noise analysis and the Bochner-Minlos Theorem within a continuous time setting, Elliott and van der Hoek (2000) and Helge et al., (1996) these control-theoretic techniques of can be generalized and applied over Hilbert spaces to accommodate infinitely divisible distributions, including long-memory fractal Brownian motion and Lévy processes[12].

Marinacci (1999) outlines a set of behavioural considerations that might motivate an approach to decision-making predicated on uncertainty aversion, while in Epstein and Schneider (2001), an axiomatic basis for uncertainty aversion has been constructed deploying a discrete-time, multiple-priors, recursive utility framework. A continuous-time variant is discussed in Chen and Epstein (2000). Also, see the debate between Epstein and Schneider (2001), and Hansen et al. (2001) over the precise nature of the relationship holding between risk-sensitive penalty functions and multiple-priors forms of generalized utility. Significantly, Grant and Quiggin have shown how Epstein and Zhang's (2001) definition of 'ambiguous events' can be used to define ambiguity aversion over preference relations in "a solely preference-based and model-free manner" (Grant and Quiggin, 2002, p. 2).

From an evolutionary (though, necessarily, somewhat speculative) perspective it stands to reason that animals would evolve a neuronal capacity for the perception of multi-fractal patterns and power-law processes within nature as this would support various anticipative and calculative forms of cognition. However, apart from a

[12] Another related body of literature concerns fractional diffusion processes and fractional calculus. For an overview see Scalas (2005), Gorenflo, Mainardi and Scalas (2004), and Podlubny (1999).

heightened capacity for pattern recognition, the presence of non-extensive informatic properties would conceivably give rise to uncertainty aversion of the kind associated with Choquet Expected Utility Theory and risk-sensitive control theory in a multiple-priors context. The sensitivity of penalty functions based on the Tsallis-distribution to fractal phenomena (for an image detection example see Piasecki et al., 2002) support the notion that *s*-shaped distortion functions and other forms of uncertainty aversion would provide animals an evolutionary cognitive advantage in environments where the stochastic processes governing relevant risks are fat tailed and conform in a fractal manner to power law distributions.

Post-Keynesian economists such as David Dequech (2000) have argued that the distinction between fundamental (Keynesian) uncertainty and ambiguity should be based on the underlying distinction between the potential unknowability or knowability of currently incomplete information. Yet according to the above arguments, non-extensivity in financial processes derives from both ontological and epistemic considerations and from interactions between each of these. In his own research on liquidity preference Juniper (1995) has suggested that, from the perspective of risk-sensitive control theory, the conventionally applied 'stochastic uncertainty constraint' governing observation error, model uncertainty, and external perturbation can be viewed as a reflection of incomplete knowledge that may either be potentially *knowable* or *unknowable in perpetuity*. In each case the formal representation would be the same. This distinction between knowability or unknowability would obviously be preserved in any generalisation of control, filtering and estimation techniques; including those that replace quadratic penalty functions and root-mean-squared measures of uncertainty with (risk-sensitive) exponential penalties and Boltzmann-Gibbs entropic measures of uncertainty (i.e. where these are captured by the difference between free and bound entropy), and ultimately those drawing on *q*-exponential penalty functions and Tsallis entropy-based measures of stochastic uncertainty. Significantly, Tsallis and Stariolo (1995) have already considered an extension of this kind in their paper on *q*-generalised of techniques simulated annealing.

6.0 A Keynesian Perspective on Conventional Finance Theory

Current asset pricing models separate portfolio decisions from those made about production and physical investment. For example, the well-known Lucas tree mode treats the dividend process as exogenously determined. When asset-pricing models are combined with stochastic growth models, the latter are usually predicated on implausible neoclassical foundations (like those of Real Business Cycle theory) where the real forces of (marginal) productivity and thrift are ultimately responsible for driving the dividend process (Brock, 1982). In such cases neither unemployment nor underutilization of capacity can arise, other than as a temporary departure from the steady-state growth path (e.g. as certain variables jump instantaneously, overshooting or undershooting to keep the economy on a rational expectations trajectory which reflects fully anticipated, though longer-term adjustment costs).

More significantly, the typical representative agent framework implicitly precludes the operation of any Keynesian *fallacy of composition* effects that might otherwise result in an insufficiency of effective demand. Similarly, long-run monetary neutrality is often implicitly guaranteed by block recursive structure of neoclassical synthesis models (see the discussion of this feature in Sargent, 1979). All of this, of course, completely ignores the extensive literature on the capital debates and the implications this debate has for how processes of economic growth should be modelled. In a multi-sectoral world where capital is acknowledged to be a reproducible good, the rather quaint notion that the marginal productivity of aggregate capital or labour could explain income distribution and growth must be abandoned. Needless to say, this realization, has implications for the econometric estimation of production functions, and undermines much of the Old and New Growth Theory—in both its stochastic or non-stochastic variants—except for those models which have merely reproduced earlier discoveries on the part of Sraffa, von-Neumann and Leontieff (Salvadori, 2003).

Juniper (2005) argues that the concept of uncertainty aversion closely corresponds to Keynesian notion of liquidity preference. While authors such as Dow and Werlang (1992) cite Frank Knight rather than Keynes, it is important to appreciate the critical differences between these two theorists. Ultimately, Knight believed that uncertainty aversion arises due to an inability on the part of certain individuals to specify the state space governing risk. Those who possess this ability are more likely to succeed in business enterprises. However, those who do not possess it themselves will not recognize this ability. Accordingly, it will be untraded, giving rise to the problem of a missing market. In contrast, Keynes adopted a more ontologically grounded view of uncertainty as something that pertains to long-term decision-making. The main ontological basis for this uncertainty is the phenomenological reality of human freedom and the creative ability to intervene in history, so transforming the nature of economic institutions and processes.

Like his predecessors—Keynes and Harrod—Hyman Minsky also embraced the *instability principle*, which is predicated on the notion that economic instability is an *endogenous* phenomenon. In Minsky's version of events, periods of optimism are seen to give rise to behavior that, in more conservative times, might appear reckless: banks, households, and firms embrace more fragile financial positions, in the sense that (present value) break-even times for investment and points of turn-around in debt-redemption are increasingly deferred. Initially, this recklessness occurs at a time when existing rates of interest are relatively small, primarily due to low levels of liquidity preference. For example, expanding firms rely more on external sources of finance rather than on retained earnings. In general, each class of agents becomes more exposed to less diversified sources of income and to financial obligations that are more rigid and inflexible. As the whole economy becomes more and more vulnerable to adverse changes in interest rates or downturns in effective demand, liquidity preference begins to rise, perversely feeding into the very process that determines the structure of short term interest rates.

Juniper (2005) has argued that a Minksyian analysis of financial instability would require an interweaving of epistemic and ontological variables. From an epistemic perspective uncertainty aversion reflects a greater sensitivity on the part of agents to the heightened consequences of any adverse movement in the spectrum of liquidity premia, these consequences, in turn, are (ontologically) determined by real changes in balance-sheet structures of economic agents: banks, households and firms. In a risk-sensitive, stochastic optimal control setting, as we have seen, each of these interwoven factors has a clear interpretation: in risk sensitive control theory, uncertainty aversion is represented by the parameter in the penalty function determining where agents are situated along the spectrum between H_2 and H-infinity control; whereas financial instability would be accommodated by an expansion in the stochastic uncertainty constraint representing model uncertainty, external perturbation and observation error.

In many applications of risk-sensitive control, where penalty functions belong to the exponential family (reflecting constant absolute risk aversion), the stochastic uncertainty constraint represents the difference between free and bounded entropy. Presumably, in applications of Tsallis entropy the penalty function would conform to the power law family and the stochastic uncertainty constraint would be determined by the difference between free and bound Tsallis entropy. In a Minskian or Keynesian world, therefore, interactions between financial institutions, firms and households are seen to be crucial. In particular, uncertainty aversion or changes in liquidity preference would directly influence the decisions that firms make about *real* (physical) investment, not just the decisions that investors make about *financial* investment. Minsky's work has spawned a variety of attempts to model financial instability using tools of non-linear dynamic simulation and analysis (Taylor & O'Connell, 1985; Foley, 1997; Keen, 1995, 1999, 2000; Chiarella and Flaschel, 2000). However, little of this analysis has spilled over to influence quantitative finance theory.

Needless to say, this interdependence between decisions of banks, households and firms has grown in importance due to the privatisation of social security, increased financial investment by middle-class households (not least through occupational superannuation); attempts by governments to fund of contingent liabilities through creation of funds leveraged over private sector activity, and an increasing reliance at all levels of government on pro-cyclical investment by private sector via private-public partnerships.

7.0 Conclusions

The literature on multi-fractals, Tsallis distributions, and anomalous diffusion processes is growing rapidly. New applications of Tsallis entropy to decision-making are occurring on a weekly basis. In this context, it is inevitable that many of the issues described within this paper and often discussed in speculative terms, will become the focus of formal analysis and detailed empirical research.

While uncertainty aversion is recognized as fundamental determinant of financial investment, it is less well appreciated that uncertainty aversion can readily be extended to non-financial investment through Real Options theory. There is a growing recognition of market incompleteness amongst Real Options theorists, because the risks applying to non-financial investment make replication difficult. This necessitates the application of either utility maximising principles or the determination of equivalent martingale measures using relative entropy or Esscher transforms that are, themselves dual to utility functions. It is thus only a short step from here to recognition of the need to apply Choquet utility theory and other forms of uncertainty aversion in real options theory, as recognised in the literature on environmental sustainability (Basili, 1998).

This step having been made, the macroeconomic implications for finance theory are that variations in uncertainty aversion will now influence the dividend process itself, via multiplier effects spreading from investment to overall levels of effective demand and aggregate activity. Fluctuations in investment are the main culprits in explaining movements in the point of effective demand, though the heightened responsiveness of consumer sentiment to developments in the markets for financial assets and property is of increasing concern to regulatory authorities. Each of these two sources of fluctuations, in turn, is primarily driven by variations in uncertainty aversion. Hysteresis effects would then influence the actual long-run rate of growth. This makes financial processes far more complex than those predicated either on exogenous dividend streams (the Lucas tree model) or those associated with stochastic growth models (Brock, 1982). And Keynesian insights into the nature of financial markets can no longer be precluded from investigation on erroneous ontological grounds. However, this paper has also been motivated by the conviction that research into aspects of this broad set of economic and financial phenomena, including uncertainty aversion, would no doubt benefit greatly from the insights and tools developed within non-extensive statistics.

Another important question is why power law distributions and scale-invariance arise in financial processes. In physics, new interpretations of quantum mechanics have shown how the Schröedinger equation can be derived within Newtonian mechanics through the imposition of scale invariance and non-differentiability (Nottale, 1995). However, these fractal properties of space and time only become pertinent at cosmological or sub-atomic scales. In finance theory and economics, scale invariance obtains at scales far removed from these extremes. However, the relativity (linear homogeneity) of all prices and the arbitrariness of choice entailed by any choice of numeraire, points to one possible avenue of interpretation (see Hoogland & Neumann). From this perspective, issues raised by Piero Sraffa's attempt to solve Ricardo's problem of finding a standard of value that would be invariant to changes in income distribution, which is nowadays usually approached through a reworking of the Perron-Fröbenius theorems, come to the fore. Andrew's Wittgensteinian insight into Sraffa's position suggests that Sraffa was all too aware of the impossibility of this task. No standard could possibly be invariant to changes such crucial factors as income distribution, the rate of technological change, the presence of increasing or decreasing returns to scale in particular industrial sectors, and heterogeneity of labour. It is this very stumbling block, which has kept Marxist analysts chasing their tails for the last two decades. At the same time, the work of Tsallis and others on social networks highlights the empirical validity of power-law scaling in network effects (Abe and Suzuki, 2003). No doubt, the multi-fractal nature of news arrival processes may contribute to the generation of multi-fractal price distributions.

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