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Deliberations on Different Forms of Fractional Calculus

James Juniper[1]

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Centre of Full Employment and Equity
The University of Newcastle, Callaghan NSW 2308, Australia
Home Page: http: //e1.newcastle.edu.au/coffee
Email: coffee@newcastle.edu.au

1. Introduction

1.1 Fractal Phenomena and Fractional Brownian Motion

Different forms of calculus have evolved rapidly in recent years motivated by the requirement to explain turbulent and fractal processes of both natural and social origins. Examples of the former include rain patterns, river flooding, wind turbulence in canopies, and a variety of quantum mechanical phenomena. Examples of the latter include internet communications, network effects and the behaviour of financial asset prices. Multi-fractal processes of anomalous diffusion have also been identified in statistical mechanics.

In previous work (Juniper, 2005, 2006) the inter-relationship between generalised information measures such as Tsallis entropy, Coherent risk measures, the distortion measures used in the actuarial sciences, the phenomenon of non-extensivity in statistical mechanics and decision-making under uncertainty, has been highlighted.

The objective of this paper is to review a variety of calculus-based approaches to the analysis fractional processes, including conventional fractional calculus, the F^{α}-calculus, the generalised Jackson calculus, and the Wick-Ito calculus are investigated. Where relevant, the specific eigenfunctions, Laplace transforms, and solutions methods associated with the specific type of calculus are discussed. In this section of the paper, Fractional Brownian Motion is defined. Then fractal random walks are introduced as a bridge to a subsequent discussion of fractional differentiation and integration. Section 1.2 examines applications of the standard fractional calculus. Section 2 considers alternative approaches: respectively the the Wick-Ito, F^{α}-, and generalised Jackson calculi. Concluding observations follow in Section 3.

A FBM (Fractional Brownian Motion) process, $\{B_H(t)\}_{t \geq 0}$, has the following properties:

1. uniqueness
2. H-self-similarity (a stochastic process $\{X(t)\}_{t \geq 0}$ is self-similar if $\{X(ct)\} \approx \{kX(t)\}$ and is H-self-similar if $k = c^H$, where H is a known Hurst parameter)
3. it has stationary increments ($E[(B_H(t+h)\text{-}B_H(h))(B_H(s+h)\text{-}B_H(h))]$
    a. if $H = ½$ it has independent increments
    b. if $H > ½$ it has long-range dependence (i.e. $\Sigma\text{Cov } B_H(1)\text{-}B_H(n+1)\text{- } B_H(n)) = \infty$)
    c. if $H \neq ½$ it is non-Markovian and not a semi-Martingale
    d. the covariance between future and past increments is positive (negative) if $H > (<) ½$.

1.1.2 The Fractal Random Walk

One way to approach such phenomena is to begin with the fractal random walk. Define a shift operator B such that its operation on a discrete data set Y shifts the index by one unit.

$$BY_j = Y_{j-1}.$$

A simple random walk can now be written as (Picozzi &West, 2002: 3),

$$(1 - B)Y_j = \xi_j .$$

This can be generalised by considering the fractional difference equation,

$$(1-B)^{\varepsilon} Y_j = \xi_j$$
$$\Rightarrow Y_j = (1-B)^{\varepsilon} \xi$$

For $|\varepsilon| < 1$, this can be written using the binomial expansion,

$$Y_j = \sum_{k=0}^{\infty} \binom{-\varepsilon}{k} (-1)^k B^k \xi_j = \sum_{k=0}^{\infty} \binom{-\varepsilon}{k} (-1)^k \xi_{j-k} \ .$$

Using identities amongst gamma functions the binomial coefficients can be written as,

$$\binom{-\varepsilon}{k} = \frac{\Gamma(1-\varepsilon)}{\Gamma(k+1)\Gamma(-\varepsilon-k+1)} = (-1)^k \frac{\Gamma(k+\varepsilon)}{\Gamma(k+1)\Gamma(\varepsilon)}.$$

In contrast to the standard random walk, now the memory extends infinitely far back in time. The strength of the influence of remote fluctuations depends on the magnitude of the binomial coefficients. In cases where ε is an integer the gamma functions have simple poles and the binomial coefficient vanishes after $\varepsilon + 1$ time-steps. Using Stirling's approximation for gamma functions it can be shown that,

$$\frac{\Gamma(k+\varepsilon)}{\Gamma(k+\beta)} \propto k^{\varepsilon-\beta}, \quad k >> \varepsilon, \beta.$$

Accordingly, for $k \to \infty$, the coupling strength becomes (Picozzi & West, 2002: 3),

$$(-1)^k \binom{-\varepsilon}{k} \propto \frac{k^{\varepsilon-1}}{\Gamma(\varepsilon)},$$

thus decreasing with increasing time-lag as an inverse power law.

Letting $\varepsilon = H - ½$, the spectrum becomes,

$$S(\omega) \approx \frac{1}{\omega^{2H-1}}; \quad \omega \to 0.$$

That is, persistence obtains For $1 \geq H > ½$, while for $½ \geq H > 0$, the spectrum increases as a power law (anti-persistence).

1.1.3 The Continuous Time Analog

The continuous version of this fractional difference equation is,

$$D_t^{\alpha}[Y(t)] = \xi(t); \quad 0 < \alpha \leq 1.$$

Rearranged into integral form, the equation can be written explicitly in terms of the Riemann-Liouville fractional integral (to be explicated below),

$$Y_{\alpha}(t) \equiv D_t^{-\alpha}[\xi(t)] = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\xi(\tau) d\tau}{(t-\tau)^{1-\alpha}}.$$

If $\xi(t)$ is a δ–function correlated Gaussian process, the system response will be Gaussian but with a variance that increases as a power law in time as given by,

$$\sigma_H^2(t) = \sigma_H^2 t^{2H}; \quad H = \alpha - \tfrac{1}{2}; \quad \sigma_H^2 = \frac{\langle \xi^2 \rangle}{2H\Gamma(H+\tfrac{1}{2})^2}.$$

By induction it can be established that the nth derivative of the function $f(t)$ is given by:

$$f^{(n)}(t)=\frac{d^n f}{dt^n}=\lim_{h\to 0}\frac{1}{h^n}\sum_{r=0}^{n}(-1)^r\binom{n}{r}f(t-rh).$$

Now let,

$$\begin{bmatrix} p \\ r \end{bmatrix}=\frac{p(p+1)\ldots(p+r-1)}{r!}.$$

Then,

$$\binom{-p}{r}=\frac{-p(-p-1)\ldots(-p+r-1)}{r!}=(-1)^r\begin{bmatrix} p \\ r \end{bmatrix}.$$

Thus, if:

$$f_h^{(p)}(t)=\frac{1}{h^p}\sum_{r=0}^{n}(-1)^r\binom{p}{r}f(t-rh),$$

then,

$$f_h^{(-p)}(t)=\frac{1}{h^p}\sum_{r=0}^{n}\begin{bmatrix} p \\ r \end{bmatrix}f(t+h)\text{, for } p \text{ a positive integer.}$$

Under the assumption that $n\to\infty$ as $h\to 0$, we can take $h=(t-a)/n$ and consider the following limit (n.b. in what follows, notationally, the left and right hand subscripts determine the interval of integration or differentiation which is read from left to right):

$$\lim_{\substack{h\to 0\\ nh=t-a}} f_h^{(-p)}(t)={}_aD_t^{-p}f(t).$$

By induction, it can be established that [46]:

$${}_aD_t^{(-p)}f(t)=\lim_{\substack{h\to 0\\ nh=t-a}} h^p\sum_{r=0}^{n}\begin{bmatrix} p \\ r \end{bmatrix}f(t-rh)=\frac{1}{(p-1)!}\int_a^t(t-\tau)^{p-1}f(\tau)d\tau$$

That this expression represents the p-fold integral can be established through integration of the relationship (Podlubny, 1999: 47):

$$\frac{d}{dt}\left({}_aD_t^{-p}f(t)\right)=\frac{1}{(p-2!)}\int_a^t(t-\tau)^{p-2}f(\tau)d\tau={}_aD_t^{-p+1}f(t),$$

from a to t to obtain:

$${}_aD_t^{-p}f(t)=\int_a^t\left({}_aD_t^{-p+1}f(t)\right)dt,$$

$${}_aD_t^{-p+1}f(t)=\int_a^t\left({}_aD_t^{-p+2}f(t)\right)dt\text{, etc.,}$$

so that,

$$ {}_aD_t^{-p} f(t) = \int_a^t dt \int_a^t dt \cdots \int_a^t f(t)dt \,, $$

where the integration is carried out *p* times.

Podlubny establishes that for derivatives of arbitrary order (Podlubny, 1999: 52-55),

$$ {}_aD_t^p f(t) = \lim_{\substack{h\to\infty \\ nh=t-a}} f_h^{(p)}(t) = \sum_{k=0}^{m} \frac{f^{(k)}(a)(t-a)^{-p+k}}{\Gamma(-p+k+1)} + \frac{1}{\Gamma(-p+m+1)} \int_a^t (t-\tau)^{m-p} f^{(m+1)}(\tau)d\tau, $$

under the assumption that the derivatives $f^{(k)}(t)$, $k = 1,2,\ldots, m + 1$ are continuous in the closed interval $[a, t]$ and that m is an integer number satisfying the condition $m > p - 1$. This is the Grünwald-Letnikov version of the fractional derivative. Riemann-Liouville Fractional Derivatives (represented below by the bold **D**) are defined by viewing the above expression as a particular case of the following integro-differential equation (Podlubny, 1999: 62),

$$ {}_a\mathbf{D}_\mathbf{t}^\mathbf{p} f(t) = \left(\frac{d}{dt}\right)^{m+1} \int_a^t (t-\tau)^{m-p} f(\tau)d\tau, \qquad m \le p < m+1. $$

$$ = \sum_{k=0}^{m} \frac{f^{(k)}(a)(t-a)^{-p+k}}{\Gamma(-p+k+1)} + \frac{1}{\Gamma(-p+m+1)} \int_a^t (t-\tau)^{m-p} f^{(m+1)}(\tau)d\tau. $$

Podlubny confirms that the operators of fractional differentiation commute (Podlubny, 1999: 59-62).

If $0 \le m < p < m+1$ and $0 \le n < q < n+1$, for $f^{(k)}(a) = 0, \quad k = 0,1,\ldots, r-1;$

and for ${}_aD_t^p$ and ${}_aD_t^q$,

$$ {}_aD_t^q\left({}_aD_t^p f(t)\right) = {}_aD_t^p\left({}_aD_t^q\right) = {}_aD_t^{p+q} f(t). $$

A similar compositional expression also holds for Riemann-Liouville derivatives (Podlubny, 1999: 74). The $k - n$th derivative of a function $f(t)$ can be written as (Podlubny, 1999: 65),

$$ f^{(k-n)}(t) = \frac{1}{\Gamma(n)} D^k \int_n^t (t-\tau)^{n-1} f(\tau)d\tau, \qquad n \ge 1,\ k \ge n. $$

Podlubny (1999: 63-4) derives this equation from the Cauchy formula, itself established through induction. Here, D^k denotes k iterated integrations. From this equation he establishes the property (Podlubny, 1999: 65-7),

$$ {}_a\mathbf{D}_\mathbf{t}^{-\mathbf{p}}\left({}_a\mathbf{D}_\mathbf{t}^{-\mathbf{q}} f(t)\right) = \frac{1}{\Gamma(p+q)} \int_a^t (t-\xi)^{p+q-1} f(\xi)d\xi = {}_a\mathbf{D}_\mathbf{t}^{-\mathbf{p}-\mathbf{q}} f(t), $$

through direct substitution and simplification. Moreover, he establishes that (Podlubny, 1999: 70-1),

$$ {}_a\mathbf{D}_t^p\left({}_a\mathbf{D}_t^{-p} f(t)\right) = f(t), $$

$$ {}_a\mathbf{D}_t^{-p}\left({}_a\mathbf{D}_t^{p} f(t)\right) = f(t) - \sum_{j=1}^{k}\left[{}_a\mathbf{D}_t^{p-j} f(t)\right]_{t=a} \frac{(t-a)^{p-j}}{\Gamma(p-j+1)}, \text{ and} $$

$$ {}_a\mathbf{D}_t^{-p}\left({}_a\mathbf{D}_t^{q} f(t)\right) = {}_a\mathbf{D}_t^{q-p} f(t) - \sum_{j=1}^{k}\left[{}_a\mathbf{D}_t^{q-j} f(t)\right]_{t=a} \frac{(t-a)^{p-j}}{\Gamma(1+p-j)} $$

for $0 \le k-1 \le q < k$.

Unfortunately, the Riemann-Liouville approach leads to initial conditions containing the limit values of the Reimann-Liouville fractional derivatives at the lower terminal $t = a$. Although such problems can be solved mathematically, there is no known physical interpretation for such initial conditions (Podlubny, 1999: 78). Accordingly, M. Caputo proposed an alternative definition of the fractional derivative,

$$ {}_a^C D_t^\alpha = \frac{1}{\Gamma(n-\alpha)} \int_\alpha^t \frac{f^{(n)}(\tau)d\tau}{(t-\tau)^{\alpha+1-n}}, \quad n-1 < \alpha < n. $$

For the Caputo derivative the following relationship obtains (Podlubny, 1999: 81),

$$ {}_a^C D_t^\alpha\left({}_a^C D_t^m f(t)\right) = {}_a^C D_t^{\alpha+m}, \quad m = 0,1,2,\ldots; n-1 < \alpha < n. $$

The differentiation operators on the left-hand side are interchanged for the Riemann-Liouville derivative.

1.2 Applying the Fractional Calculus

1.2.1 The Fractional Derivative

Hartley and Lorenzo (2002), define the fractional-order integral using the Riemann-Louisville fractional integral,

$$ {}_a\mathbf{D}_t^{-q} x(t) \equiv \frac{d^q x(t)}{\left[d(t-a)^q\right]} \equiv \int_a^t \frac{(t-\tau)^{q-1}}{\Gamma(q)} x(\tau)d\tau, \quad t \ge a. $$

Likewise, the derivative is given by:

$$ {}_a\mathbf{D}_t^{q} x(t) \equiv \frac{d^m}{dt^m}\left({}_a d_t^{q-m} x(t)\right), \quad t \ge a $$

They begin with the initialised fractional-order operator defined as:

$$ {}_c\mathbf{D}_t^{-q} x(t) \equiv {}_c d_t^{-q} x(t) + \psi(x, -q, a, c, t), \quad t > a $$

where ψ, the initialisation function, which accounts for the effects of the past is given by,

$$ \psi(x, -q, a, c, t) \equiv \int_a^c \frac{(t-\tau)^{q-1}}{\Gamma(q)} x(\tau)d\tau, \quad t \ge c. $$

1.2.2 Laplace Transforms for Fractional Calculus

In engineering and signal-processing approaches to the control and estimation of systems of ordinary differential equations, the use of Laplace transforms is popular. The Laplace transform of a system of first order linear differential equations can be written as a linear matrix equation in powers of the complex Laplace transform variable *s*. Equations of this kind can then be solved using standard techniques of linear and polynomial algebra followed by an application of the inverse Laplace transform. Hartley and Lorenzo base their straightforward analysis of fractional-order systems on the analysis of transfer functions using Laplace transforms. Given a fundamental linear fractional-order differential equation:

$$ {}_c\mathbf{D}_t^q x(t) \equiv {}_c d_t^q x(t) + \psi(x,q,a,c,t) = -ax(t) + bu(t), \;\; q > 0 $$

Hartley and Lorenzo assume that $c = 0$, and that $\psi = 0$ temporarily, to give:

$$ {}_0 d_t^q x(t) = -ax(t) + bu(t), \;\; q > 0 , $$

before applying the Laplace Transform, resulting in:

$$ s^q X(s) = -ax(s) + bU(s), \;\; q > 0 $$

$$ \frac{X(s)}{U(s)} = F(s) = \frac{b}{s^q + a}, \; q > 0 $$

Thus, the resulting system transfer function can thus be expressed as follows:

This transfer function, which the authors identify as the fundamental building block of more complicated fractional-order systems, is not contained in standard Laplace transform tables. However, they note that the following transform pair is commonly available:

$$ \frac{1}{s^q} = L\left\{\frac{t^{q-1}}{\Gamma(q)}\right\} $$

The system transfer function can be expanded term-by-term to derive a generalised impulse response function Hartley and Lorenzo call the *F*-function. Ignoring the *b* term in the numerator, the system transfer function is expanded about s = ∞, using long division:

$$ F(s) = \frac{1}{s^q + a} = \frac{1}{s^q} - \frac{a}{s^{2q}} + \frac{a^2}{s^{3q}} \cdots = \frac{1}{s^q} \sum_{n=0}^{\infty} \frac{(-a)^n}{s^{nq}}, \;\; q > 0 $$

$$ L^{-1}\{F(s)\} = \frac{t^{q-1}}{\Gamma(q)} - \frac{at^{2q-1}}{\Gamma(2q)} + \frac{a^2 t^{3q-1}}{\Gamma(3q)} - \cdots, \;\; q > 0 $$

Term-by-term transformation of this expression using the inverse operator from the original Laplace transform pair yields:

Accordingly, by collecting the right-hand side into a summation, it can be seen that the resulting Generalised Impulse Response Function is given by:

$$ F_q[-a,t] \equiv t^{q-1} \sum_{n=0}^{\infty} \frac{(-a)^n t^{nq}}{\Gamma((n+1)q)}, \;\; q > 0 $$

Which is thus associated with the following Laplace Transform Identity:

$$ L\{F_q[a,t]\} = \frac{1}{s^q - a}, \;\; q > 0 $$

Lorenzo and Hartley observe that F_q[-a,t] is a generalised exponential, as can be shown by setting the *q* parameter to unity:

$$F_1[-a,t] = \sum_{n=0}^{\infty} \frac{(-at)^n}{\Gamma(n+1)} \equiv e^{-at}$$

Having established the requisite tools, the authors turn to the Scalar Initialization Problem. The Laplace transform is applied to the fundamental equation (with *c* set equal to 0) to yield:

$$L\left\{ {}_0\mathbf{D}_t^q x(t) \right\} = s^q x(s) + L\left\{ \psi(x,q,a,0,t) \right\}, \ \forall q \in \Re$$

The second term appearing on the RHS of the above expression is now given by:

$$L\left\{ \psi(x,q,a,0,t) \right\} = L\left\{ \frac{d}{dt} \left[\frac{1}{\Gamma(1-q)} \int_a^0 \frac{x(\tau)}{(t-\tau)^q} d\tau \right] \right\}$$

$$\Rightarrow X(s) = \frac{b}{s^q + a} U(s) - \frac{1}{s^q + a} \psi(s)$$

This expression can then be inverse transformed using the Laplace convolution theorem to yield:

$$x(t) = \int_0^t F_q[-a,t] Bu(t-\tau) d\tau - \int_0^t F_q[-a,\tau] \psi(x,q,a,0,t-\tau) d\tau$$

Here, the first term represents any forced response due to $u(t)$, while the second term represents the initialisation response of the system to the past history of $x(t)$.

Hartley and Lorenzo go on to consider general vector representations and their solutions, as well as fractional vector initialisation, feedback, and vector estimators, while the sinusoidal response of fractional-order operators and PID control are also examined.

1.2.3 The Fractional Derivative and its Eigenfunction

Solutions to fractional partial differential equations are arrived at by using the Laplace transforms of the Riemann-Louisville fractional integral and fractional derivative and their Caputo counterparts (see Podlubny, 1999: 105-6). However, an alternative and, in many ways more straightforward, approach is taken by Lorenzo and Hartley (1999; 2000), who begin by asking the question: What is the eigenfunction for fractional differointegration? The answer that they provide is the *R*-Function[1]:

$$R_{q,v}[a,c,t] \equiv \sum_{n=0}^{\infty} \frac{(a)^n (t-c)^{(n+1)q-1-v}}{\Gamma((n+1)q-v)}, \ t > c$$

Lorenzo and Hartley show that this function subsumes a variety of other generalised functions appearing in the literature on fractional calculus, including their own $F_q[.]$ function, Erdelyi's generalised Mittag-Leffler function (see Podlubny, 1999), and

[1] An alternative approach is afforded by Rocco and West (1998), who demonstrate that the fractional derivative (integral) of increments to the complex Generalized Weierstrass Function, which is defined by:

$$W(t) = \sum_{n=-\infty}^{\infty} \frac{\left(1 - e^{i\gamma^n t}\right) e^{i\phi_n}}{\gamma^{(2-D)n}},$$

where $1 < D < 2$, $\gamma > 1$, and ϕ_n is an arbitrary phase, is another fractal function with a greater (lesser) fractal dimension.

Miller-Ross's (1993) function. Applying the Reimann-Louisville derivative to this function results in the following:

$$ {}_{c}\mathbf{D}_t^q R_{q,0}\left(a,c,t\right) = \sum_{n=0}^{\infty} \frac{\left(a\right)^n {}_{c}\mathbf{D}_t^q \left(t-c\right)^{(n+1)q-1}}{\Gamma\left(\left(n+1\right)q\right)} = \sum_{n=0}^{\infty} \frac{\left(a\right)^n \left(t-c\right)^{nq-1}}{\Gamma\left(nq\right)} $$

As, ${}_{C}\mathbf{D}_t^q \left[x-a\right]^p = \dfrac{\Gamma\left(p+1\right)\left[x-a\right]^{p-v}}{\Gamma\left(p-v+1\right)}$

Lorenzo and Hartley (1999: 7) establish the eigenfunction property of the *R*-function under fractional differentiation (for the parameterization of $v = 0$):

$$ {}_{c}\mathbf{D}_t^q R_{q,0}\left(a,c,t\right) = \left(a\right) R_{q,0}\left(a,c,t\right) + a \lim_{m\to -1} \frac{\left(a\right)^m \left(t-c\right)^{(m+1)q-1}}{\Gamma\left(\left(m+1\right)q\right)} $$

$$ \therefore\ {}_{c}\mathbf{D}_t^q R_{q,0}\left(a,c,t\right) = aR_{q,0}\left(a,c,t\right),\quad t > c,\ \ q > 0; $$

For $a = 1$ and $v = 0$, the *R*-function returns itself. Under *u* order differintegration the *R*-function returns another *R*-function:

$$ {}_{c}D_t^u R_{q,v}\left[a,c,t\right] = R_{q,(v+u)}\left[a,c,t\right]. $$

Hartley and Lorenzo (1999) show that the *R*-function specialises to the exponential function, sine cosine, hyperbolic sine and hyperbolic cosine functions. These results engender a series of trignometric identities, which are further discussed in Lorenzo and Hartley (2000).

The Laplace transform of the the *R*-function is given by (Lorenzo and Hartley: 2000: 4):

$$ L\left\{R_{q,v}\left(a,c,t\right)\right\} = \frac{e^{-cs} s^{v}}{s^{q} - a},\quad c \ge 0,\ \mathrm{Re}[(n+1)q - v] > 0,\ \mathrm{Re}\ s > 0. $$

2. Alternatives to the Standard Fractional Calculus

2.1. The Wick-Ito Calculus and Finance Theory

The non-Markovian and non-semi-martingale properties of FBM make it hard to construct the integration of FBM[2]. Moreover, pathwise integration,

$$ \int_a^b f\left(t,w\right)\delta B_H\left(t\right) = \lim_{|\Pi|\to 0} \sum_{k=0}^{n-1} f\left(t_k, w\right)\left(B_H\left(t_{k+1}\right) - B_H\left(t_k\right)\right), $$

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[2] Fokker-Planck partial differential equations are usually derived from the Chapman-Kolmogorov equation for a Markov process. However, it is also possible to derive the Fokker-Planck equation from an Itô stochastic differential equation (Frieman, 1975). However, McCauley (2007) observes that finitely many states of memory are allowed in Kolmogorov's two partial differential equations (backward and forward). McCauley (2007) articulates the precise relationship holding between Kolmogorov's two pde's and the Chapman-Kolmogorov equation for Itô processes with finite memory. In McCauley et al (), it is further established that martingale stochastic processes generate uncorrelated but generally *nonstationary* increments. A detrended process with a drift dependent on the state variable is generally not a martingale. Although martingales might appear Markovian when analysed using simple averages and two-point correlations, long term memory can be detected and possibly exploited at three-point or higher correlations.

where Π is a partition of the interval [*a*, *b*] and $|\Delta| = \max_{0 \le k \le n-1} (t_{k+1} - t_k)$, fails to produce an arbitrage market due to misbehaviour of the Gaussian kernel near 0 (Daye, 2003; citing Rogers, 1997). This is why the Norwegian and Chinese 'white-noise mafias' have felt obliged to develop a Wick-Ito calculus defined over Hilbert spaces[3]. This paper will go into much detail in regard to this difficult domain of quantitative finance, which demands a great deal in mathematical terms from its proponents.

Suffice it to say that a separable Hilbert space, $L^2_\phi(\Re)$ (i.e. one featuring countable yet dense subsets) is constructed characterised by an inner product operator ϕ. An isometry is established between this Hilbert space and the conventional $L^2(\Re)$ space (Hu and Oksendal, 2000: lemma 2.1). A new extended Schwartz space is constructed over which a probability measure can be guaranteed using the Bochner-Minlos theorem (Holden et al., 1996: Theorem 2.1.1 and Appendix A). The integration of functions is defined using the resulting probability measure. The Kolmogorov-Centsov theorem then guarantees the existence of a *t*-continuous version of the FBM so defined. Using Hermite polynomials, an orthonormal basis is established for the separable Hilbert space $L^2_\phi(\Re)$. Using the fractional Wiener-Itô chaos expansion theorem, the Wick-Ito integral is then constructed over this ortho-normal basis (Hu et al, Theorem 2.6). The associated Wick algebra of product, convolution, power and exponential operations (the latter defined by analogy using convergence to the Wick analog of a conventional power series), supports a Wick calculus, which enables the development of Wick-Itô directional and stochastic gradients, and conditional expectations of functions and integrals. Fractional versions of the Ito rule and the Girsanov formula can be derived to constitute what is required for a complete fractional stochastic calculus, which can then be applied in financial modelling and asset pricing of FBM processes (Hu and Oksendal, 2000; Necula, 2002). Using the Wick algebra and calculus, no arbitrage and completeness can be defined for admissible and self- financing portfolios. A fractional version of the Black-Sholes formula can be derived through the application of the fractional version of Girsanov's theorem, along with a fractional Black-Sholes Partial Differential Equation (Necula, 2002: Theorem 4.3). The conventional and newly-derived fractional PDEs differ from one another in accordance with the following substitution:

$$\sigma \leftrightarrow \sqrt{2H}\sigma t^{H-\frac{1}{2}} .$$

As for its Brownian Motion counterpart, through manipulation of the fractional Black-Sholes PDE the usual 'Greek alphabet' of risk measures (Δ, Γ, Θ, V, and ρ) can be derived for options priced over the FBM process (Necula, 2002: Theorem 5.4).

The main drawback of the Wick-Ito calculus, despite its obvious power, is the complexity of the Hilbert space constructions upon which it is based. Undoubtedly, this will serve to limit the extent of its application compared to other, more user-friendly, rivals. However, as Parvate & Gangal (2003: 2) argue, the development of a local fractal calculus based on Riemann integration has value due to the transparency and constructive nature of such an approach: one that also possesses some algorithmic advantages.

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[3] A similar trajectory is followed in Decreusfond and Üstünel (1999), who also use the Malliavin calculus in a stochastic calculus of variations setting to obtain the requisite Itô formula, Itô-Clark representation formula and Girsanov theorem for the functionals of a FBM.

2.2 The F^α Calculus

2.2.1 The Mass and Staircase Functions

Operators for the local fractal F^α-calculus are constructed by first defining a mass function as a replacement for the length of the interval, which possesses the properties of interval-wise additivity, translation and power law (Parvate and Gangal, 2005: 392).

For a subdivision $P[a,b]$ of the interval $[a, b]$, $a < b$, the mass function $\gamma^\alpha(F, a, b)$ is given by:

$$\gamma^\alpha(F,a,b) = \lim_{\delta\to 0} \inf_{\{P_{[a,b]}:|P|\le\delta\}} \sum_{i=0}^{n-1} \frac{(x_{i+1}-x_i)^\alpha}{\Gamma(\alpha+1)} \theta(F,[x_i,x_{i+1}]),$$

where $\theta(F, x_i, x_{i+1}) = 1$ if $F \cap [x_i, x_{i+1}]$ is non-empty and zero otherwise, and

$$|P| = \max_{0\le i\le n-1}(x_{i+1}-x_i),$$

the infinum being taken over all subdivisions P of $[a, b]$ such that $|P| \le \delta$.

The authors observe (Parvate and Gangal, 2005: 392) that the motivation for this mass function comes from fractional calculus (specifically the $1/\Gamma(\alpha + 1)$ term, which appears in both the Riemann-Louiville and Caputo definitions of the fractional derivative) and the construction of the Hausdorff measure. For the purposes of this paper it is instructive to observe that the α-power term performs the same role as the parameter measuring the degree of non-extensivity in non-extensive statistical mechanics. Juniper (2005) follows Tsallis (1995) in arguing that, this non-extensivity parameter also represents the degree of uncertainty aversion in a decision-making context.

The integral staircase function—a generalisation of the Lebesgue-Cantor staircase function—can easily be obtained from this mass function. Let α_0 be arbitrary but fixed real number. The integral staircase function $S_F^\alpha(x)$, is then given by (Parvate and Gangal, 2005: 392):

$$S_F^\alpha(x) = \begin{cases} \gamma^\alpha(F,\alpha_0,x), & \text{if} \quad x \ge \alpha_0 \\ -\gamma^\alpha(F,\alpha_0,x) & \text{otherwise} \end{cases}$$

The integral staircase function has the following properties (Parvate and Gangal, 2005: 393):

1. $S_F^\alpha(x)$ is increasing in x.
2. If $F \cap (x, y) = \varnothing$, then $S_F^\alpha(x)$ is a constant in $[x, y]$.
3. $S_F^\alpha(y)$ - $S_F^\alpha(x)$ = $\gamma^\alpha(F, x, y)$
4. S_F^α is continuous on (a, b).

For the F^α-derivative and integral operators, the change in the staircase function over an interval replaces the length of the interval [x, y]:

$$(y-x) \to S_F^\alpha(y) - S_F^\alpha(x).$$

The staircase function is continuous and increasing in the interval. The F^α-derivative is defined over an α-perfect set, H, which represents the minimal closed set amongst the class of sets giving rise to the same staircase function. It is constructed by introducing the notion of the point of change of a function *f*: the pint *x* is a point of change of the function if *f* is not constant over any interval (*c*, *d*) containing *x*. The set of all such points is called the set of change and is denoted by Sch*f*. Let $F \subset \Re$ be such that $S_F^\alpha(x)$ is finite for all $x \in \Re$ for some $\alpha \in (0, 1]$. Then the set $H = \mathrm{Sch}(S_F^\alpha)$ is said to be α-perfect. It is the case that (Parvate and Gangal, 2005: 393):

1. $S_H^\alpha = S_F^\alpha$
2. if $x \in H$ and $y < x < z$ then either $S_H^\alpha(y) < S_H^\alpha(x)$ or $S_H^\alpha(x) < S_H^\alpha(z)$.
3. for any point $x \in H$, there is at most one more point $y \in H$ such that $S_H^\alpha(x) = S_H^\alpha(y)$.

2.2.2 The F^α Fractional Integral

The principles of Riemann integration are followed in constructing the integral, although the Riemann-Stieltjes sum is now defined over increments in the staircase function. For members of the class of functions $f: \Re \to \Re$, which are bounded on *F*, denoted as $f \in B(F)$, the particular function *f* is F^α-integrable if (Parvate and Gangal, 2005: 396):

$$\underline{\int_a^b} f(x)\,d_F^\alpha x = \overline{\int_a^b} f(x)\,d_F^\alpha x,$$

where the upper and lower bounds are defined as follows:

$$\underline{\int_a^b} f(x)\,d_F^\alpha x = \sup_{P_{[a,b]}} L^\alpha[f, F, P]$$

and,

$$\overline{\int_a^b} f(x)\,d_F^\alpha x = \inf_{P_{[a,b]}} U^\alpha[f, F, P]$$

with,

$$L^\alpha[f, F, P] = \sum_{i=0}^{n-1} m[f, F, [x_i, x_{i+1}]]\left(S_F^\alpha(x_{i+1}) - S_F^\alpha(x_i)\right)$$

and,

$$U^\alpha[f, F, P] = \sum_{i=0}^{n-1} M[f, F, [x_i, x_{i+1}]]\left(S_F^\alpha(x_{i+1}) - S_F^\alpha(x_i)\right)$$

and where,

$$m[f, F, P] = \begin{cases} \inf_{x \in F \cap I} f(x) \text{ if } F \cap I \neq \phi \\ 0 \quad \text{otherwise} \end{cases},$$

and,

$$M[f, F, P] = \begin{cases} \sup_{x \in F \cap I} f(x) \text{ if } F \cap I \neq \phi \\ 0 \quad \text{otherwise} \end{cases}$$

for a closed interval *I*.

The F^{α}-integral of f on $[a, b]$ is denoted by:

$$\int_a^b f(x) d_F^{\alpha} x$$

and is given by the common value for the upper and lower bounds defined above. The properties of the F^{α}-integral operator include linearity and, if f is an F^{α}-integrable function on $[a, b]$ with $a < b$ and $c \in [a, b]$, then f is F^{α}-integrable on $[a, c]$ and $[c, b]$ with (Parvate and Gangal, 2005: 396):

$$\int_a^b f(x) d_F^{\alpha} x = \int_a^c f(x) d_F^{\alpha} x + \int_c^b f(x) d_F^{\alpha} x .$$

Moreover, if $\chi_F(x)$ is the characteristic function of $F \subset \Re$, then,

$$\int_a^b \chi_F(x) d_F^{\alpha} x = S_F^{\alpha}(b) - S_F^{\alpha}(a).$$

Let $F \subset \Re$, $f\colon \Re \to \Re, x \in F$. A number l is said to be the limit of f through the points of F, or simply the F-limit as $y \to x$ if given any $\varepsilon > 0$, there exists a $\delta > 0$, such that (Parvate and Gangal, 2005: 394):

$$y \in F \text{ and } |y - x| < \delta \Rightarrow |f(y) - l| < \varepsilon .$$

If such a number exists it is denoted by,

$$l = F - \lim_{y \to x} f(y).$$

Similar constructions apply in defining the notion of F-continuity. If F is an α-perfect set, then the F^{α}-derivative of f at x is given by (Parvate and Gangal, 2005: 396):

$$D_F^{\alpha}(f(x)) = \begin{cases} F - \lim_{y \to x} \dfrac{f(y) - f(x)}{S_F^{\alpha}(y) - S_F^{\alpha}(x)} & \text{if } x \in F \\ 0 & \text{otherwise} \end{cases},$$

if the limit exists. The properties of the resulting F^{α}-derivative include that it is a linear operator, the derivative of a constant function is zero, and that the derivative of the integral staircase function is the characteristic function. Parvate and Gangal (2005: 397) establish two theorems that are analogous to the fundamental theorems of calculus:

The analog to the first fundamental theorem obtains if $F \subset \Re$ is an α-perfect set, and $f \in B(F)$ is an F-continuous function on $F \cap [a, b]$, and,

$$g(x) = \int_a^x f(y) d_F^{\alpha} y$$

for all $x \in [a, b]$. Then (Parvate and Gangal, 2005: 397):

$$D_F^{\alpha} g(x) = f(x) \chi_F(x).$$

The analog to the second fundamental theorem applies in the case where $f\colon \Re \to \Re$ is a continuous, F^{α}-differentiable function such that Sch(f) is contained in an α-perfect set F, and $h\colon \Re \to \Re$ is an F-continuous function such that:

$$h(x)\chi_F(x) = D_F^\alpha f(x).$$

Then,

$$\int_a^b h(x) d_F^\alpha x = f(b) - f(a).$$

Finally, Parvate and Gangal (2005: 397) introduce an F^α-version of the Taylor's series expansion. Under conditions of F^α-differentiability, a function H has a Taylor series expansion:

$$h(w) = \sum_{n=0}^{\infty} \frac{\left(S_F^\alpha(w) - S_F^\alpha(x)\right)^n}{n!} \left(D_F^\alpha\right)^n h(x).$$

Recognizing that e^x is eigenfunction of derivative operator dy/dt = y, this raises the question of what the operator is for which the F^α-derivative is the eigenfunction. Parvate and Gangal (2005: 398) consider the equation:

$$D_{F,t}^\alpha x = \chi_F(t) A x.$$

Here, A is an $n \times n$ constant matrix. The solution to this equation for $x \in \Re$ is given by:

$$x(t) = \exp\left[S_F^\alpha(t) A\right] x_0.$$

2.2.3 Conjugacy Relations between the F^α- and ordinary Riemann-calculus

Let J = [0, 1] and let C = middle 1/3 Cantor set. Then one can associate with every point $x \in J$ a point of C and vice versa through the map f: $J \rightarrow C$. Assume that $x \in J$ is represented in binary notion, as say $x = 0.x_1x_2\ldots$ while points $y = 0.y_1y_2\ldots$ of C are represented in base 3 by {0, 2}, with all middle third parts removed (i.e. the 1 digits). Let g: {0, 1} $\rightarrow$ {0, 2} so that $f(x) = f(0.x_1x_2\ldots) = y = 0.y_1y_2\ldots$ with $y_i = g(x_i)$ and $f^{-1}(y) = x = 0.x_1x_2\ldots$. Parvate and Gangal call f: $J \rightarrow C$ the fractalizing map and f^{-1} : $C \rightarrow J$ the defractalizing map. The other operators are defined as follows: for an F^α-integrable function defined over [a, b], let,

$$f_1(x) = \int_a^b f(x') d_F^\alpha x', \quad x \in [a, b]$$

through defractalization a function $g = \psi[f]$ is obtained which is Riemann integrable over $\left[S_F^\alpha(a), S_F^\alpha(b)\right]$.

Moreover, denoting:

$$g_1(y) = \int_{S_F^\alpha(a)}^{y} g(y') dy', \;\; y \in \left[S_F^\alpha(a), S_F^\alpha(b)\right],$$

then Parvate and Gangal (2005: theorem 15, Appendix A) establish that:

$$g_1\left(S_F^\alpha(x)\right) = f_1(x), \quad x \in [a, b].$$

This can be summarised by the relation,

$$I_F^\alpha = \phi^{-1} I \psi,$$

where I^{α}_{F} is the F^{α}-integration operator, I is the indefinite Riemann integration operator, and ϕ is just a restriction of ψ to a smaller class of functions (i.e. those differentiable on $\left[S_F^{\alpha}(a), S_F^{\alpha}(b)\right]$

A similar conjugacy relationship holds for F^{α}-derivatives (Parvate and Gangal, 2005: 401). Under certain conditions, if a certain function h is F^{α}-differentiable at x, then the function $g = \phi[h]$ obtained by defractalization, is differentiable (in the ordinary sense) at $y = S_F^{\alpha}(x)$. Further, the following holds:

$$\left.\frac{dg}{dy}\right|_{y=S_F^{\alpha}(x)} = D_F^{\alpha}\left[h(x)\right].$$

If the ordinary differential operator is represented as D, then it is the case that:

$$D_F^{\alpha} = \phi^{-1} D \phi .$$

2.3 The Jackson Calculus

2.3.1 Tsallis Entropy and the *q*-calculus

Tsallis entropy is defined by:

$$S_q\left(p_1, p_2, \cdots p_W\right) = \frac{1}{1-q}\left[\sum_{i=1}^{W}\left(p_i\right)^q - 1\right]\left(q > 0\right).$$

Under equiprobability:

$$S_q = \max S_q = k \ln_q W, \quad \text{where } \ln_q x \equiv \frac{x^{1-q} - 1}{1-q} \text{ and } \ln_1 x = \ln x.$$

The inverse function of $\ln_q x$, called the q-exponential (Tsallis et al, 1998, p. 537), is given by,

$$e_q^x \equiv \exp_q(x) \equiv \left[1 + (1-q)x\right]^{1/(1-q)} \quad \text{with } e_1^x = e^x .$$

Suyari (2004: 2) defines the q-product as follows,

$$x \otimes_q y := \begin{cases} \left[x^{1-q} + y^{1-q} - 1\right]^{\frac{1}{1-q}}, & \text{if } x > 0, y > 0, x^{1-q} + y^{1-q} - 1 > 0 \\ 0, \text{ otherwise} & \end{cases}$$

The q-product is derived by requiring that is satisfy the following equations:

$$\ln_q\left(x \otimes_q y\right) = \ln_q x + \ln_q y$$
$$\exp_q(x) \otimes_q \exp_q(y) = \exp_q(x + y)$$

The continuous time version of Tsallis entropy is given by,

$$S_q \equiv \frac{1 - \int dx[p(x)]^q}{q-1}, \quad q \in \Re; \quad S_1 = -\int dx p(x) \ln p(x), \quad x \in \Re^d$$

For $q = 1$, Tsallis entropy reverts to the familiar Boltzmann-Gibbs entropy. Under appropriate moment constraints over the first and second moments of the distribution, Boltzmann-Shannon entropy can be used to derive the familiar Gaussian process.

However, under slightly modified moments constraints (which take into account the divergence of the second moment), De Souza and Tsallis (1997) also show that Tsallis entropy can be used to derive the Students-*t* distribution.

When Tsallis entropy is maximised subject to the following moment constraints over the *q*-generalised mean and *q*-generalised variance,

$$\langle x \rangle_q = \bar{\mu}_q = \int x \frac{\left[p(x)\right]^q}{\int \left[p(x)\right]^q dx} dx; \quad \left\langle \left(x-\bar{\mu}_q\right)^2 \right\rangle_q = \bar{\sigma}_q^2 = \int \left(x-\bar{\mu}_q\right)^2 \frac{\left[p(x)\right]^q}{\int \left[p(x)\right]^q dx} dx$$

the resulting distribution is the *q*-Gaussian PDF as given by[4],

$$f(x) = \frac{\exp_q\left(-\beta_q x^2\right)}{\int \exp_q\left(-\beta_q x^2\right) dx}, \ \beta_q > 0.$$

The *q*-exponential $\exp_q(\pm\lambda t)$ emerges as the solution of a variety of differential equations in non-extensive statistical mechanics including:

$$\frac{d}{dt}\left\{\left[1 \pm (1-q)\lambda t\right] f(t)\right\} = \pm(2-q)\lambda f(t).$$

2.3.2 The Tsallis Distribution and Jackson Derivative

Borges (2004) has derived a generalised version of the Jackson Derivative based on principles of reasoning by analogy. Borges points out that e^x is eigenfunction of derivative operator dy/dt = y. He then asks, 'What is the operator for which *q*-exponential is the eigenfunction?' He demonstrates that the answer lies in the following operator [5]:

$$D_{(q)} f(x) \equiv \lim_{y\to x} \frac{f(x)-f(y)}{x \oplus_q^{-1} y} = \left[1-(1-q)x\right] \frac{df(x)}{dx}$$

The counter-part to this generalisation of the Jackson derivative is the following integral operator:

$$\int_q f(x) d_q x = \int \frac{f(x)}{1+(1-q)x} dx, \quad d_q x = \lim_{y\to x} \frac{1}{1+(1-q)x} dx$$

Lenzi et al (1999) generalise the Laplace transform using the *q*-exponential and *q*-logarithmic functions. The usual exponential kernel exp(-*st*) is replaced by the *q*-

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[4]Suyari and Tsukuda (2005) shows that the Tsallis distribution can also be derived by taking the maximal value of the *q*-product of the likelihood function, $L_q(\theta)$, shown below,
$L_q(\theta) = L_q(x_1, x_2, ..., x_n; \theta) := f(x_1-\theta) \otimes_q f(x_2-\theta) \otimes_q \cdots \otimes_q f(x_n-\theta)$ at
$\theta = \theta^* := \frac{x_1 + x_2 + \cdots + x_n}{n}$.

[5] The $\oplus^{-1}$ function stands for the inverse operation to the *q*-sum, as shown below,

$$x \oplus_q y \equiv x + y + (1-q)xy$$

$$x \oplus_q^{-1} y \equiv \frac{x-y}{1+(1-q)y}, \quad y \neq 1/(q-1)$$

Borges (2004) shows how these *q*-generalized operators follow necessarily from asking what kind of 'sum' operator would be required to make the exponential terms in the product of two *q*-exponentials 'add' together.

Gaussian kernel $[\exp_q(-t)]^s$. Properties of the resulting q-Laplace transform are discussed (which include linearity and scaling along with a variety of q-Laplace transform pairs, while applications in the field of non-extensive statistical mechanics are considered.

2.3.3 Diffusion Equations

Tsallis (2005) shows how the Fokker-Planck equation for normal diffusion (satisfying $\langle x\rangle^2 \propto t$), which governs the heat equation (also known to be the diffusion equation characterising Einsteinian Brownian motion),

$$\frac{\partial p(x,t)}{\partial t} = D\frac{\partial^2 p(x,t)}{\partial x^2} \quad (D>0);$$

can be generalised in two different ways. The first form is linear but has non-integer or fractional derivatives,

$$\frac{\partial p(x,t)}{\partial t} = D\frac{\partial^\gamma p(x,t)}{\partial x^\gamma} \quad (D>0;\ 0<\gamma<2).$$

The second form preserves integer derivatives but is non-linear,

$$\frac{\partial p(x,t)}{\partial t} = D\frac{\partial^2 [p(x,t)]^\nu}{\partial x^2} \quad (\nu \in \Re).$$

While the first of these forms (under the initial condition $p(x, t) = \delta(x)$) gives rise to Lévy distributions, the second (under the same initial condition) yields the Tsallis distribution[6]. Tsallis posits a yet more general version of the Fokker-Planck equation,

$$\frac{\partial p(x,t)}{\partial t} = D\frac{\partial^\gamma [p(x,t)]^{2-q}}{\partial |x|^\gamma} \quad (\delta, \gamma, q \in \Re).$$

At this level of generality, the diffusion equation does not possess analytical solutions. However, by constructing a q-generalised version of the Fourier transform using q-products and the q-exponential, Umarov, Tsallis, Gell-Mann and Steinberg (2006a,b) accomplish a q-generalization of the central limit theorem (which pertains to q-Gaussian distributions) and its Lévy-Gnedenko counterpart (which pertains to q-generalised Lévy distributions). The latter limit theorem would seem to characterise stochastic processes conforming to Tsallis' most generalised version of the diffusion process. Nevertheless, at this stage, it would appear that there are no sufficiently generalised versions of stochastic calculus that would point to a solution for the most general version of the Fokker-Planck equations listed above, although the Wick-Ito calculus would seem to point in the direction of a likely solution.

The first form of the duffusion process is amenable to analysis using conventional fractional calculus, The F^α-calculus, or the Wick-Ito approach. The second form can obviously be analysed using the generalised Jackson calculus.

3. Conclusion

This paper has reviewed four varieties of fractional calculus. From a practitioner perspective, Harltey and Lorenzo's approach to the fractional calculus is

[6] For an extensive discussion of the relationship between this diffusion equation, Hurst exponents, power-law scaling, and Markov processes see Bassler et al., (2006).

straightforward and can readily be incorporated into existing Laplace transform-based methods of estimation and control. The Wick-Ito calculus offers the greatest generality but demands a great deal of the practitioner in mathematical terms. The Jackson calculus is also congruent with the existing Laplace transform-machinery. However, it is specialised to particular varieties of anomalous diffusion.

The attractions of the F^{α}-construction are that it also affords the prospect of bridging the more familiar fractional (non-integer) though linear non-local calculus and the integer though non-linear Jackson calculus. This is because power-law processes would seem to be accommodated by the non-linearity, which first appears in the denominator of the mass function. As mentioned in the introduction, this power term is related to the principle of non-extensivity in the literature on both statistical mechanics and generalised information measures, and also represents the degree of uncertainty aversion in a decision theoretic context (Juniper, 2005, 2006).

The conjugacy relationship established between ordinary differential equations and fractional differential equations supports both an intuitive appreciation of the calculus and an algorithmic approach to analysis. This conjugacy and the resulting ability to solve F^{α}-differential equations by fractalizing, solving using ODEs, then defractalizing, may also provide an alternative justification for neural networks and support vector machines to that afforded by regularization theory or the representing kernel Hilbert space theorems in machine learning.

The F^{α}-calculus may also prove to be congruent with the q-generalised algebra and trigonometry underpinning the Gell Mann, Tsallis, Umarov and Steinberg limit theorems for q-generalised Gaussian and Lévy processes. In particular, a more general version of the Laplace transform may be constructed by combining the q-exponential form with Hartley and Lorenzo's R-Function version of the Laplace transform. Alternatively, the conjugacy relations underpinning the F^{α}-calculus could conceivably be combined in some way with the generalised Jackson calculus. In addition, more work needs to be done to relate Martingale theory and Ito's Lemma to this new calculus (and its generalised Jackson calculus counterpart), so that it might eventually displace the far more complex Hilbert space-based machinery of the Wick-Ito Calculus.

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[1] The author is Research Associate, Centre of Full Employment and Equity, University of Newcastle.